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Changing Mathematics Learning through Changing Teachers' Thinking

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CHANGING MATHEMATICS LEARNING
THROUGH
CHANGING TEACHERS' THINKING

A Thesis Presented

by

PAMELA JOY COOKE

Submitted to the Office of Graduate Studies
and Research of the University of Massachusetts
at Boston in partial fulfillment of the
requirements for the degree of

MASTER OF ARTS

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Critical and Creative Thinking Program

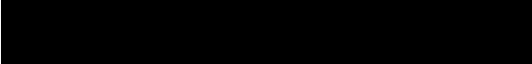
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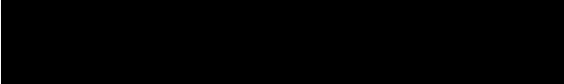
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I am also very grateful to Arthur Millman and John Murray, my committee members, for their time and thoughtful comments.

I dedicate this thesis to my husband, William, who never doubted for a moment that I could do it, and to Zachary, who always slept at the right times.

ABSTRACT
CHANGING MATHEMATICS LEARNING
THROUGH
CHANGING TEACHERS' THINKING

MAY 1991

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In the context of the goals for reform in mathematics education, as advocated by the National Council of Teachers of Mathematics, this thesis calls for elementary level students to be actively engaged in learning mathematics through the use of hands-on materials and problem solving situations which involve investigation, reasoning, and communication. These mathematical goals are discussed and then related to the more general critical thinking skills of identifying and formulating questions, asking and answering questions, investigating and analyzing data, deducing and judging deductions, inducing and judging inductions, defining terms, and interacting with others. This thesis is based heavily on the experience of the author, as she evolved from being a traditional elementary mathematics teacher, novice student of critical and creative thinking, and skeptical

participant in her first Mathematics a Way of Thinking workshop to becoming a confident and thinking mathematics teacher, flexible and effective workshop leader, and strong advocate for reform in mathematics education.

From these experiences, it has become clear to the author that in order for the goals for reform to be met, there must not only be changes in what is taught, but also in how it is taught. In order for teachers to change the way they teach, they must re-learn mathematics in a framework that involves them in active learning and small group interaction with an instructor who models strategies and behaviors for teaching thinking. In this thesis, the author shares her experiences in trying to become this type of teacher trainer.

This thesis examines the Mathematics a Way of Thinking workshop as a model for effective teacher training and provides sample mathematical lessons as instruments for change. Ten teachers who participated in the author's workshops and who are trying to implement change in their own classrooms were interviewed. Dialogues with these teachers are quoted to indicate their experiences of change in the learning and teaching of mathematics.

Three factors emerge as being most important to all of these teachers: use of manipulative materials, a supportive classroom environment, and the resulting view of mathematics as a sense-making process. These factors are consistent with the current goals for reform in mathematics education for students. In order for the advocated reform to occur, teachers and students alike must experience the creative, open-ended side of mathematics, enjoy the process of doing mathematics within a supportive framework, and develop increased confidence in their ability to think mathematically.

TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGEMENTS.....	iv
ABSTRACT.....	v
LIST OF FIGURES.....	xii
Chapter I.	
INTRODUCTION: A SEARCH FOR EFFECTIVE	
MATHEMATICS TEACHING.....	1
The Seeds of Discontent.....	1
Search for Solutions.....	4
The Urge to Spread the Word.....	7
Research: What Makes a Difference?.....	8
How Information was Collected.....	9
What Emerged from the Interviews.....	10
Overview of Remaining Chapters.....	12
Chapter II.	
CRITICAL THINKING IN MATHEMATICS EDUCATION.....	15
The Call for Change in Mathematics	
Education.....	15
The Need for Change.....	15
The Goals of Change.....	18
What is Critical Thinking in	
Mathematics?.....	20
A List of Critical Thinking Skills for	
Mathematics.....	21
Critical Thinking Dispositions.....	28
Factors That Enhance Learning to Think	
in Mathematics.....	31
Manipulative Materials.....	31
As models for an algorithm.....	32
Tools for investigation.....	33
Aids to problem solving.....	34
Manipulatives as as Aid to	
Critical Thinking.....	35
A Supportive, Facilitative	
Environment.....	36
Organizing the classroom.....	37
Developing an atmosphere	
of trust and respect.....	38
Demonstrating that thinking	
is valued.....	40
Mathematics as Sense-making.....	44
Traditional School Mathematics.....	44
Making Sense of Mathematics.....	47

CHAPTER III.

CHANGING HOW TEACHERS TEACH MATHEMATICS.....	51
The Challenge.....	51
Why Teachers Teach as They Do.....	52
How Teachers Learned	
Mathematics.....	53
Results of Learning by Absorption....	55
Beliefs about mathematics.....	55
Feelings about mathematics.....	57
Beliefs about teaching.....	60
The New Role for the Teacher	
of Mathematics.....	63
Teacher Behaviors.....	64
Teaching as Thinking.....	65
Analysis of teaching	
and learning.....	65
Teaching as problem solving....	69
Teaching as a Critical Thinking	
Activity.....	71
The key to change.....	72
Retraining Teachers: Effective Approaches..	74
Willingness to Change.....	75
Changing Knowledge.....	76
Knowledge of mathematics.....	77
Knowledge of teaching	
mathematics.....	80
Affective Change.....	82
Conclusion.....	86

CHAPTER IV.

DESCRIPTION OF THE WORKSHOP.....	89
Philosophy, Goals, and Structure.....	89
The Mathematics a Way of Thinking	
Philosophy.....	89
Goals of the Workshop.....	90
Manipulative Materials and Active	
Learning.....	93
Place Value and Regrouping.....	93
Area of a Triangle.....	99
Multiplication of Fractions.....	101
Probability.....	103
Manipulatives in Non-Routine	
Problem Solving.....	106
Supportive Learning Environment.....	108
Working with Others.....	109
Outcomes of Working with Others.....	111
Role of the Instructor in Creating	
the Environment.....	112

Open-ended questions.....	113
Questions of clarification or challenge.....	114
Questions that invite speculation, investigation....	114
Questions that ask for generalizations, conclusions, and reasons.....	115
Instructor Responses.....	116
Wait time.....	117
Accepting responses.....	117
Clarifying responses.....	118
Facilitative responses.....	119
An example of questioning and responding.....	119
Empathy of Instructor.....	121
Implications for Sense-making.....	123

CHAPTER V.

TEACHERS' REACTIONS TO THE WORKSHOP.....	127
The Process of Collecting Information.....	127
The Interviews.....	127
What Emerged from the Interviews.....	128
Effect on Teachers of Using Manipulative Materials.....	129
New Insights into Old Concepts.....	130
Insights into Implications for Teaching Children.....	131
Effect of a Supportive Environment.....	134
Shedding Anxiety.....	135
Security through Working with Others..	137
Insights into Teaching.....	138
Overall Effect on Beliefs and Attitudes about Mathematics.....	139
Changed Beliefs About Mathematics.....	140
New Flexibility.....	141
New Confidence.....	144
Effect on Approach to Teaching.....	145
Questioning.....	146
Goals for Students.....	148
Control.....	150
Teaching as Thinking.....	152
Conclusion.....	156

CHAPTER VI.	
ISSUES AND REFLECTIONS.....	160
The Curriculum.....	161
Testing.....	163
Administration.....	164
Parent Expectations.....	166
What's Next?.....	167
Knowledge of Mathematics.....	168
Teaching Thinking.....	169
Sense of Collegiality.....	170
Time for Change.....	172
Reflections on My Own Growth.....	174
SELECTED BIBLIOGRAPHY.....	182
APPENDIX A	187
APPENDIX B	191

LIST OF FIGURES

	<u>Page</u>
1. Base six trading board and recording strip.....	95
2. Trading board for addition and subtraction.....	97
3. Multiplying fractions on the geoboard.....	103
4. Possible sums with two dice.....	106
5. Tangram chart.....	107

C H A P T E R I

INTRODUCTION: A SEARCH FOR EFFECTIVE MATHEMATICS TEACHING

This thesis is about changing the way elementary school teachers teach mathematics, so that it becomes a dynamic, exciting subject in which teachers and students alike can explore both the power of mathematics and the power of their own thinking. In this chapter I shall describe my own relearning of mathematics through the use of concrete materials and a problem solving, critical thinking approach, and the consequent effect that experience has had on how I think about and teach mathematics. I shall explain how my work as a trainer of other teachers in this process led me to be interested in determining what aspects of the training have had an impact on their thinking and teaching. I shall describe how I gathered information, and my formulated conclusions.

The Seeds of Discontent

I was an elementary school teacher for about fifteen years, most recently in grade four. As with most teachers at the elementary level, I taught everything, and tried to do it all well. For the first several years that I taught, I didn't think much about teaching mathematics; I followed the book and the basic model presented by the teachers with whom

I had studied. I talked and the students listened (or did they?); I explained rules and conducted drill and practice exercises; I checked the papers for the right answers, and puzzled over how students could possibly fail something that was so clear and factual. In my math classroom, there were lots of rules, lots of facts, lots of worksheets, and lots of boredom. When time was short, or something disrupted the day, math was always the first thing to go. I was bored with it, too.

Through an extensive inservice program offered in my school system several years ago, when I was teaching second grade, I became familiar and comfortable with the process approach to learning science, and convinced of how much both the enthusiasm of my students and their achievement soared in response to an approach that put them in charge of their own learning and thinking. Over a period of three years I was co-author of a book of interdisciplinary activities for grades K to six that emphasized the development of process skills. For the following three years, as a member of a system-wide enrichment team, I participated in ongoing training that focused on techniques for teaching critical and creative thinking skills. I then enrolled in the Critical and Creative Thinking Program at the University of Massachusetts. Throughout my first year of coursework there, I gained more confidence and expertise in practicing a teaching approach that infused thinking into the curriculum; except in math.

None of my training to this point in the process approach to teaching or in critical thinking had dealt specifically with mathematics. Yet because my approach to teaching was increasingly one in which I urged my students to seek reasons and meanings, I began to be increasingly dissatisfied with how I was teaching mathematics. I realized that not only were my students not 'getting' what they were doing, but that I lacked any depth of understanding that might help me use other strategies to help them understand. I myself did not understand, beyond a very superficial level, many of the things I was supposed to be teaching to my students; yet I was expected to have all the answers. And from the students I sensed, always, the underlying question: "Why are we doing this?"

In my search for alternatives, I had read and heard a little about using manipulative materials and a more exploratory approach to teaching mathematics. It seemed to make sense in the context of my growing understanding about how children learn. But reading about and knowing how are very different. All I knew about math was what I had learned and how I had learned it. In spite of my training in teaching for thinking, I couldn't transfer those ideas to mathematics. I couldn't imagine how one could have, for instance, an open-ended question in math, because math, to me, was just facts to be memorized.

Through a weekend workshop my school system sponsored, I had a brief exposure to Mathematics Their Way, a manipulative based program for grades K-2 offered by the Center for Innovation in Education in Campbell, California. I began using some of the ideas in my second grade classroom, feeling my way for how to make it fit the textbook, which I still viewed as the ironclad curriculum. The activities I tried, mostly with pattern and place value, worked, in that they captured the students' interest, and definitely engaged them in thinking about what they were doing. And they captured my interest in learning more about the approach.

Search for Solutions

Knowing that I would be teaching grade four the next fall, I sought out a Mathematics a Way of Thinking workshop the next summer. Mathematics a Way of Thinking, also offered by the Center for Innovation in Education, is a manipulative based approach for teachers of grades 3-6. The workshop is an intense thirty hour course, given over a week. It included: an exploration of several math content areas using a variety of hands-on materials; working with small groups to explore concepts, gather data, and solve problems; and sharing solutions to problems done outside of class. The instructor modeled management techniques for materials, group work, and teaching strategies that made it clear that we were expected to think and question, and that our thinking was

valued. There were no answers given by the instructor; the meaning and the answers that we derived from the activities, the materials, and our attempts to solve the problems presented to us came from our own thinking. At the beginning of the week this was unnerving. I was angry and frustrated. Where was "the" answer; on whose authority could I take my answer, or a classmate's, to be the correct one? However, by the end of the week I had a sense of confidence about my ability to learn to do mathematics that I had never had before, and a different understanding of what mathematics was all about.

The Mathematics a Way of Thinking workshop dramatically altered my own ideas about mathematics. For the first time I began to understand some very basic concepts, such as place value, probability, logical thinking, and why you often get a smaller number when you multiply two fractions. I understood finally that formulas are derived from patterns in mathematics, rather than by magic. I also began to recognize where my mathematical anxieties came from, where my weaknesses lay, and also that I did have some strengths. I realized that my definition of mathematics had been limited to numbers and operations, namely arithmetic, and that mathematics was much more broadly defined, and included exciting and thought provoking ideas. Through the classwork and homework, I discovered that there were many kinds of problem solving, and many strategies and ways of thinking

about problems. At the end of the week, I realized that what I was coming away with was not just a collection of activities that I could use to help me begin teaching fourth grade, but an approach to teaching mathematics in a broad sense, that went far beyond 'the book', and that was compatible with my commitment to teaching for thinking.

There were many changes in my approach to teaching math as a result of my experience in the workshop. I began introducing concepts through concrete materials whenever possible, and in ways that would be relevant to my students. My goal became for my students to understand and apply concepts, rather than memorize steps to get an answer. The questions that I asked did not ask for answers so much as exploration of ideas, explanation of thinking, and sharing of ideas and strategies. More and more I avoided the textbook, using it only for practice or review, and I had begun to trust my own judgements about what students understood to guide the pace and direction of lessons. I tried to develop 'real' problems from everyday situations and to include explorations into areas of mathematics that have often been left out of the lower grades, such as logic or spatial thinking. I tried to find and develop more and more connections to other curriculum areas. At one time I had struggled to fill forty minutes a day with a math lesson; now sometimes we did it all day, in some form.

The Urge to Spread the Word

During the next few years I used the Math a Way of Thinking approach in my classroom. Simultaneously, I trained in the summers to become an instructor for the workshop. Training consisted of serving a kind of apprenticeship with several different instructors. Though the same workshop outline is followed by all instructors, there are variations in how different instructors present activities and answer questions. The Center for Innovation in Education believes that the best way for trainees to find methods that work best for their own presentations is to see as many different approaches as possible. This mirrors the way that problem solving strategies are explored in the workshop itself.

Training follows a fairly predictable sequence. In the first workshop or two the trainee usually observes and assists with materials and organization. In later workshops she or he begins to present activities with the guidance of the instructor. How much the trainee teaches in a given workshop is determined by the trainee's comfort level - what she or he feels ready to do - and the instructor's assessment of their readiness and understanding of the activities and the purpose behind them. Ideally, the trainee has a final training workshop in which she or he is a co-instructor, working with an experienced lead instructor whose role is to observe the trainee teach the majority of the workshop. The

lead instructor then spends time 'debriefing' with the trainee, helping her or him fine tune where necessary. New instructors then usually co-teach, or share teaching a workshop with another instructor, until they feel ready to teach on their own. This entire training process may take two to four years, and include four to eight workshop experiences as a trainee.

Research: What Makes a Difference?

In the past four years I have taught sixteen Math a Way of Thinking workshops as a solo instructor, with an average of 35 people in each workshop. Participants hand in a daily journal card throughout the workshop, on which they comment on each day's activities, ask questions, or react in any way they wish to what we are doing. At the end of the week each participant is also asked to write an evaluation to be sent to the Center for Innovation in Education. Both the comments on the daily cards and the evaluations have been very positive overall, and have indicated to me that, as a result of the workshop, many teachers have experienced a change in attitude and a new understanding about mathematics similar to my own. I became interested in knowing in what ways the workshop experience affected others.

For purposes of this thesis, I decided to interview some past participants in the workshop with the following questions in mind: Did it change their beliefs and thinking

about mathematics? Did it change how they teach mathematics, both in terms of teaching for thinking and what they include as content? Did it give them enough confidence to become less 'book bound', and more empowered to decide for themselves what their students were ready to learn? What particular factors or aspects of the workshop structure and content had the greatest impact on those who did report such changes?

How Information Was Collected. I chose to interview ten teachers who had taken the workshop with me at several different sites, all within the last year and a half. Many of these teachers were also taking the follow-up sessions for the workshop, a series of six after-school sessions that meet throughout the school year to review the work done in the summer, expand upon it, and to provide a support group as they work to implement a new approach to teaching math. I chose teachers who had indicated by their comments that they were excited by the workshop experience, both in terms of their own new understanding of mathematics, and in terms of the possibilities they saw for classroom practice. I was interested in finding out what had brought this about for the teachers who reported such changes. I chose teachers who had a range of years of teaching experience, and who had no mathematics background or other special mathematics training besides that required for elementary teaching certification.

I purposely made the interviews very informal and open-ended. I asked each teacher why she or he had taken the workshop, and then what aspects of the workshop had had the greatest impact on her or him. My purpose was to encourage each teacher to talk in fairly general terms about the workshop, and then to look for common threads in what they reported. I occasionally asked a general question or asked the teacher to elaborate on something she or he had said. I purposely did not go into the interviews armed with a set of specific questions.

What Emerged from the Interviews. Three components emerged as important aspects of the workshop for all the teachers interviewed. These were:

1. Use of manipulative materials. Teachers reported that their own understanding of many mathematics concepts was greatly enhanced through the use of concrete materials. Some reported that for the first time they truly understood some concepts that they had been teaching for years. For many teachers, this was their first experience learning through hands-on materials, and they were very excited by their own ability to make sense of the concepts and by the potential for their students' learning.
2. Positive, supportive environment. For many

teachers, this was their first experience working on mathematics in a cooperative rather than a competitive framework. In addition to relieving pressure to perform and fear of failure, teachers reported that working with a group encouraged them to share their strengths, and gave them a sense of success. As a result of the instructor's approach, over the course of the week they began to develop a new sense of confidence in their own ability to think through mathematical situations.

3. New view of math as open-ended and sense-making.

Most elementary teachers had a math learning experience that was based entirely on rote learning and drill and practice. They came to view math as a body of facts and rules to be memorized. Though some enjoyed math because it was so black and white and predictable, for most it was a series of disconnected skills, becoming more difficult to memorize and more incomprehensible as they went up through the grades. In the workshop, the instructor encourages the participants to share many different ways to solve a problem, and does not validate one way as being the right way or even the preferable way. Some problems have more than one right answer. All ideas are accepted and investigated. Questions are open-ended, and

participants are constantly asked to explain their thinking, make connections, and find and analyze patterns. As a result, many teachers began to see mathematics as a sense-making process, and to see themselves as able to 'do' math for the first time. They no longer needed to wait for someone to tell them whether they were right or wrong, but had increasing confidence that they could determine that for themselves.

As a result of this work, I have concluded that the Mathematics a Way of Thinking Workshop, by involving teachers in relearning math in a critical thinking framework, provides an experience that can powerfully affect their beliefs and their understanding about mathematics, and that can thereby begin to change their approach to teaching mathematics.

Overview of Remaining Chapters

In Chapter II, I shall examine the need for change in mathematics education. I shall conclude that the goals of reform can be summarized as teaching students to think mathematically, and that this includes a range of critical thinking skills. I shall define critical thinking, and identify the critical thinking skills and dispositions that are included in learning to 'think mathematically'. I shall examine the three components of the Mathematics a Way of Thinking workshop that I identified as a result of my

interviews in terms of how each fits within the context of critical thinking skills.

In Chapter III, I shall examine the challenges involved in bringing about real change in how teachers teach mathematics. I shall argue that a key to this change is that teachers must relearn mathematics in the way in which we wish them to teach it. If we wish for them to teach through manipulative materials and a problem solving approach, they must be immersed in doing mathematics that way themselves. If we wish for them to teach for thinking, then they must relearn mathematics concepts using critical thinking skills, with an instructor who models what a teacher who teaches for thinking does. I shall define the role of the teacher who teaches for thinking and examine the behaviors that are characteristic of such a teacher. I shall conclude that teaching in this way involves the teacher in a diagnostic, critical thinking process her or himself. I shall identify the critical thinking skills used by a teacher teaching in this way. I shall propose that the Mathematics a Way of Thinking workshop is an effective training program which involves teachers in thinking critically in mathematics, and which models strategies and behaviors for teaching for thinking.

In Chapter IV, I shall examine the philosophy and goals of the Mathematics a Way of Thinking workshop, and describe how it incorporates manipulative materials and a

supportive environment and how participants are helped to develop a view of math as open ended and sense-making. I shall explain how the activities and the instruction combine to engage the participants in thinking mathematically, and how the instructor models teaching for thinking, as defined in Chapters II and III.

In Chapter V, I shall present and discuss comments from the teachers whom I interviewed. I shall look at how their comments reflect an impact on them in the following categories:

1. The effect of using manipulative materials to learn and/or relearn mathematics concepts.

2. The effect of a supportive environment on teachers' development of confidence about doing and teaching mathematics.

3. The effect of the workshop overall on beliefs and attitudes towards mathematics.

4. The effect of the workshop experience on teachers' approach to teaching mathematics.

In Chapter VI I shall summarize my thoughts about the Mathematics a Way of Thinking experience and the challenge of changing the teaching of mathematics in general. As a part of this, I shall reflect on my own growth and change as a result of my work with other teachers.

C H A P T E R I I

CRITICAL THINKING IN MATHEMATICS EDUCATION

The Call for Change in Mathematics Education

In this chapter I shall examine the arguments for the need for reform in mathematics education. I shall conclude that the goals of reform can be summarized as teaching students to think critically in mathematics. I shall define critical thinking, and identify the critical thinking skills that are applicable to this view of learning mathematics. I shall then examine the ways in which these skills are enhanced through use of manipulative materials, a supportive, facilitative classroom environment, and the portrayal of mathematics as an open-ended, sense-making activity.

The Need for Change. The call for change in mathematics education is widespread. It comes from researchers, educational leaders, and those in business and industry concerned with having a competent workforce. It springs from a concern for economic health, social and educational equity, and the needs of individuals dealing on a daily basis with a more and more mathematical world.

There is a consensus among these sources that the world has become increasingly technologically complex, but that mathematics education has not changed to teach students the skills they will need to cope with this complexity. As the

National Research Council (NRC) notes in Everybody Counts: A Report to the Nation on the Future of Mathematics Education, science and technology have come to influence all aspects of life, and increasingly, mathematical skills are essential to the kinds of jobs that will be a critical part of our society.

Mathematics is the key to opportunity. No longer just the language of science, mathematics now contributes in direct and fundamental ways to business, finance, health, and defense. For students, it opens doors to careers (1989, p. 1).

This assessment is reflected by Lindquist, who states that, "over 60 percent of college career choices are closed if one has not taken advanced mathematics in high school" (NRC 1989, p.3).

The National Council of Teachers of Mathematics (NCTM) in Curriculum and Evaluation Standards for School Mathematics (henceforth referred to as the NCTM Standards), addresses this issue of opportunity as one of the major reasons that mathematics education must change. School curricula now in place are a product of the industrial age, when educational goals were aimed at preparing students to work in factories and shops, and become informed voters. This educational system does not prepare students for a society and an economy in which, "information is the new capital and the new material, and communication is the new means of production" (NCTM 1989, p.3).

Many of the skills required of this new workforce are mathematical, and of a type that has nothing much to do with the kinds of skills that are taught in schools.

Traditional notions of basic mathematical competence have been outstripped by ever-higher expectations of the skills and knowledge of workers; new methods of production demand a technologically competent workforce... Businesses no longer seek workers with strong backs, clever hands, and "shopkeeper" arithmetic skills (NCTM 1989, p.3).

Arithmetic and computation have been the emphasis of the school mathematics curriculum; however, the skills needed by today's workforce far exceed arithmetic. In the Standards the NCTM summarizes these skills as follows:

the ability to set up problems with the appropriate operations;
knowledge of a variety of techniques to approach and work on problems;
understanding of the underlying mathematical features of a problem;
the ability to work with others on problems;
the ability to see the applicability of mathematical ideas to common and complex problems;
preparation for open problem situations, since most real problems are not well formulated;
belief in the utility and value of mathematics.
(NCTM 1989, p. 4)

Steen also confirms the need for a new view of essential skills:

Today's students will live and work in the 21st century, in an era dominated by computers, by worldwide communication, and by a global economy. Jobs that contribute to this economy will require workers who are prepared to absorb new ideas, to perceive patterns, and to solve unconventional problems. Mathematics is the key to opportunity for these jobs. The mathematical sciences... have become an essential ingredient in the education of all Americans. (Steen 1989, p. 18)

There is agreement that broader and better mathematics education must be the norm for all students (NCTM 1989, NRC 1989, Lindquist 1989), for economic and social reasons. Without equitable educational opportunities, the dichotomy between those who have mathematical skills and those who do not will create "an intellectual elite and a polarized society" (NCTM 1989, p. 9), in which a few have the knowledge needed to control scientific development and technology, and therefore the economy. Such a society is not consistent with the values of a democratic system or with its economic needs.

The "habits of mind" developed by studying mathematics are also seen as contributing in a broader sense to the values of a democratic system. Those who can think mathematically learn to "distinguish evidence from anecdote, to recognize nonsense, to understand chance, and to value proof" (NRC, p. 8). These are abilities that are valuable to all citizens in any age.

The Goals of Change. In order to meet the challenge of preparing citizens for the modern world, our goals for mathematics education must change. What was once considered "the basics" is now woefully inadequate. "Basic skills today and in the future mean far more than computational proficiency...Topics such as geometry, probability, statistics, and algebra have become increasingly more important, and accessible to students through technology" (NCTM 1989, p.

66). We must go far beyond the mechanical use of algorithms and memorization of facts that has been the core of school mathematics.

Students who live and work using computers as a routine tool need to learn a different mathematics than their forefathers. Standard school practice...simply cannot prepare students adequately for the mathematical needs of the twenty-first century. (Steen 1990, p. 2)

Knowledge of mathematics can no longer be viewed as a static set of facts to be absorbed. It must be seen more as a dynamic set of tools, including factual knowledge, problem solving abilities, and thinking strategies that can be applied in a wide range of contexts. New goals for mathematics education, then, must include the development of skills in areas of mathematics that have been either given short shrift or entirely left out of traditional mathematics programs in the lower grades, such as logical reasoning, probability, pattern and relationships, and open-ended problem solving (NCTM 1989).

In order to attain these goals, a primary aim of mathematics education must be that of teaching students to think critically in mathematics. In the Standards the NCTM states, "A climate should be established in the classroom that places critical thinking at the heart of instruction" (1989, p. 29). In the next section, I shall define critical thinking in mathematics, and the specific skills included.

What is Critical Thinking in Mathematics?

Ennis (1987) defines critical thinking as "reasonable reflective thinking that is focused on deciding what to believe or do" (p. 10). In his analysis of specific critical thinking skills or abilities that follow this definition, his focus is based on a context of determining the validity of arguments, and judging the truth of conclusions. The example that he uses to illustrate critical thinking skills in action is his experience as a juror at a murder trial, where truth and justice are the issues.

Beyer (1985) and Swartz and Perkins (1989) similarly define critical thinking in terms of judging the truth or worth of statements or arguments, and conclude that the function of critical thinking is primarily one of evaluation.

Swartz and Perkins, however, go on to modify their definition of critical thinking. "We should not conceive of critical thinking solely as a technique for settling the truth and justice of things, but rather as an enterprise of inquiry and understanding" (1989, p. 39). Presseisen also defines critical thinking more broadly, emphasizing its application to analyzing and evaluating arguments, but also to "generate insight...develop cohesive logical reasoning patterns..." (1985, p. 45).

These broader definitions of critical thinking, with their emphasis on skills of inquiry and reasoning which lead to understanding, is the definition that will be used in this

paper to define critical thinking in mathematics. This definition is also in keeping with the stated goals of NCTM .

In summary, the intent of these goals is that students will become mathematically literate. This term denotes an individual's ability to explore, to conjecture, and to reason logically, as well as to use a variety of mathematical methods effectively to solve problems. (1989, p. 6)

In elaborating on the goal of putting critical thinking at the heart of instruction, the Standards again stress skills of inquiry and reasoning:

Both teachers' and children's statements should be open to question, reaction, and elaboration from others in the classroom...Children need to know that being able to explain and justify their thinking is important and that how a problem is solved is as important as its answer. (p. 29)

I will use Ennis's inclusive list of critical thinking skills (see Appendix A) as the basis for defining a list of skills that form the core of critical thinking in elementary school mathematics, with a focus on inquiry and reasoning. I will also identify the critical thinking dispositions that must be cultivated in order to bring about critical thinking in this context.

A List of Critical Thinking Skills for Mathematics

Ennis's list of critical thinking abilities and dispositions (1987) is very comprehensive. He points out that it is organized so that it might be used as an outline of goals for a critical thinking curriculum or college level course. It is partly for this reason that it is so

comprehensive; he does not mean to imply that all the skills should be taught at all levels, nor that all the skills would be applicable to every subject. Rather, certain skills are more appropriate than others to certain subjects and certain levels of cognitive ability. Keeping in mind the definition that I have established for critical thinking in elementary school mathematics, I have identified the skills from Ennis's list that are important to inquiry and reasoning.

I shall define inquiry as seeking information, examining, investigating, and questioning. Critical thinking skills that pertain to inquiry are:

1. Identifying and formulating questions
2. Asking and answering questions of clarification or challenge
3. Investigating

I shall define reasoning as drawing inferences or conclusions, analyzing, thinking logically. Critical thinking skills that pertain to reasoning are:

4. Analyzing
5. Deducing and judging deductions
6. Inducing and judging inductions

In addition to these I shall include the critical thinking skills of:

7. defining terms
8. interacting with others

These are skills that are applicable to any content area, and have a definite place in mathematics.

I shall define each skill according to the general meaning given by Ennis (1987), and show how each is reflected

in goals for mathematics education established by the NCTM (1989).

1. Identifying and formulating questions. According to Ennis (1987), this skill is the first step in clarifying what it is you need to think about. Identification of a problem, especially in an unclear or "fuzzy" situation, is often the first step toward solving the problem.

The NCTM standards state that students should learn to formulate questions and problems in a variety of contexts, rather than always being presented with a clearly defined question for which they must mechanically produce a right answer. "Students... should experience problem situations rich in opportunities to formulate and define problems, determine the information required, decide on methods for obtaining this information, and determine the limits of acceptable solutions" (1989, p. 76).

2. Asking and answering questions of clarification and challenge. Ennis considers this ability so obviously important that he does not elaborate upon it, but simply gives examples of the kinds of questions that need to be asked - or answered - to clarify meanings, ideas, actions, or conclusions, such as 'Why?', 'What is an example?', 'What do you mean by _____?'. Such questions are a key step in getting to a clear understanding of a situation, a problem, or its resolution.

The NCTM Standards repeatedly stress the need for students to be asked questions that require them to justify their answers and their thinking, and to learn to ask such questions themselves. The Standards see teachers as "probing" for students' ideas. (1989, p. 10). "Consistent use of such questions as 'Why do you think that's a good answer?' or 'Do you think you would get the same answer if you used the other materials?' conveys to children the importance of critical thinking and establishes a spirit of enquiry" (p. 29).

3. Investigating. Ennis sees this as a subskill of inducing and judging inductions (pp. 13-14). He does not elaborate on this, but I conclude that he means that once one makes a generalization or a conjecture, the next step is to collect facts, evidence, or explanations to support or disprove it. Another aspect of investigation involves searching for facts or information to answer a question or to solve a problem.

The overall approach to mathematics education outlined by the Standards is investigative. The Standards specify that mathematics concepts and skills should be developed from problem situations, and refer to problem solving as "a method of inquiry and investigation" (1989, p. 75), in which students should explore the use of variables, verify and interpret results with respect to the original problem, and learn more than one way to represent and solve problems.

Group work is recommended so that students can "discuss strategies and solutions, ask questions, examine consequences and alternatives...verify results, interpret solutions, and question whether a solution makes sense" (NCTM 1989, p. 76).

4. Analyzing arguments. In Ennis's definition, this includes a composite of interrelated abilities, including identifying conclusions, determining the validity of conclusions, identifying reasons not explicitly stated, recognizing irrelevant statements, and establishing and testing criteria. (1987, p. 18) Although Ennis applies this set of skills to a situation involving the determination of guilt or innocence, its more general application is to judging the validity of any argument, including a mathematical argument.

The Standards state that students can and should learn to recognize inductive and deductive reasoning, evaluate mathematical conjectures, and validate their own thinking. Even very young children can be led by a skillful teacher to identify valid and invalid arguments and to use the language of logic in informal situations (NCTM 1989, p. 30). The Standards call for a curriculum that allows the time and experiences for students "to develop their ability to construct valid arguments in problem settings and evaluate the arguments of others" (NCTM 1989, p. 81).

5. Deducing and judging deductions. In elaborating on this skill, Ennis comments that he means to "include its

practical aspects...Basically deduction is concerned with whether something follows necessarily from something else" (p. 20). Is our conclusion consistent with all the facts we have collected?

The Standards do not propose that young students be taught formal logic; like Ennis, they also stress application of the practical aspect of this ability, stating that the study of mathematics should emphasize reasoning so that students can draw logical conclusions about mathematics and apply deductive reasoning. This is seen as a natural outgrowth of a mathematics program which encourages students to explore, conjecture, validate thinking, and convince others. "Both inductive and deductive reasoning come into play as students make conjectures and seek to explain why they are valid" (NCTM 1989, p. 81).

6. Inducing and judging inductions. In this skill category, Ennis includes generalizing and inferring conclusions and hypotheses (1987, p. 13). Identifying a pattern is often an important step toward being able to make a prediction, a generalization, or a conjecture.

The Standards state that students should learn to recognize and apply inductive reasoning, make and evaluate mathematical conjectures, generalize solutions and strategies to new problem situations, and recognize, describe, and generalize patterns and use them to make predictions of real world phenomena. The search for and analysis of pattern is

seen as a valuable tool for helping students develop inductive thinking. "Identifying patterns is a powerful problem-solving strategy. It is also the essence of inductive reasoning. As students explore problem situations...they can often consider or generate a set of specific instances, organize them, and look for a pattern" (NCTM 1989, p. 82).

7. Defining terms. Ennis states that defining terms is a key aspect of clarification of thinking (1987, p. 22). Clear exchange of ideas cannot take place if those attempting to communicate do not first establish a common understanding of concepts and terminology.

In the Standards, communication is recognized as an important aspect of learning to think mathematically. "The communication process requires students to reach agreement about the meanings of words and to recognize the crucial importance of commonly shared definitions" (NCTM 1989, p. 78).

8. Interacting with others. We are concerned here with the subskills of interacting with others that Ennis called "argumentation" (1987, p. 15). "Interacting with others in discussions, presentations, debates, and written pieces is crucial for critical thinkers" (p. 23).

The Standards state that students should have numerous opportunities for communicating about mathematics. Through small group problem solving, sharing of ideas and strategies, and through writing about mathematics, they should learn to

justify answers and solution processes, validate their own thinking to others, and make conjectures and convincing arguments. It is through such communication that students learn to clarify their thinking. "Opportunities to explain, conjecture, and defend one's ideas orally and in writing can stimulate deeper understanding of concepts and principles" (NCTM 1989, p. 78).

Critical Thinking Dispositions. Ennis states that critical thinking dispositions or attitudes are "essential" for the development of critical thinking abilities (p. 16). Without the disposition to seek reasons, for instance, one will most likely not learn the importance of asking questions of clarification. I have identified the critical thinking dispositions that are important to developing skills of inquiry and reasoning in elementary students. They are:

- a. Seek reasons
- b. Look for alternatives
- c. Be open-minded
- d. Seek as much precision as the subject will allow

I shall explain each disposition in terms of its importance to learning mathematics as a critical thinking process.

a. Seek reasons. A student with this disposition is more likely to formulate questions, to ask questions of clarification, or to investigate patterns and relationships in order to find out why they occur. A student without this disposition is more likely to passively accept learned

procedures, and to be content with knowing simply how something is done.

b. Look for alternatives. A student with this disposition will be more likely to investigate possible solutions to a problem, to analyze arguments, and to make a variety of possible conjectures that can be tested. Without this disposition, a student will tend to accept a given answer or procedure as the correct one, rather than questioning whether an answer makes sense, or whether a procedure or strategy is the best approach.

c. Be open-minded. A student with this disposition is more likely to investigate a variety of ideas for solutions to a problem, and to interpret patterns and relationships more creatively. She or he will be more likely to listen to the ideas of others, and consider those ideas in modifying her or his own approach or understanding. Without this disposition, a student is more likely to reject alternative ways of thinking about mathematics, and to remain rigid in her or his approach to doing mathematics.

d. Seek precision. A student with this disposition is more likely to persevere in her or his attempts to find solutions to non-routine and complex problems and to justify the validity of a solution once it is found. She or he wants to know the answer, and when one is found, wants to know that it is right, and why. Without this disposition, students apply procedures according to the 'one right way to one right

answer' rule; they want to get a right answer, but not necessarily to know why they are right or wrong.

Many educators and researchers besides the NCTM agree that an important goal of mathematics education must be teaching students to use critical thinking skills (Lappan and Schram 1989; Romberg, Zarinnia and Collis 1990; Kaplan, Yamamoto and Ginsburg 1989). All of these sources discuss desirable goals for mathematics education in terms of critical thinking behaviors: reasoning, seeing and analyzing relationships, making conjectures, application of criteria, and making judgements.

Mathematics is, above all else, a habit of mind that helps clarify complex situations. Students must learn to gather evidence, to make conjectures, to formulate models, to invent counterexamples, and to build sound arguments (Steen 1989, p. 19).

Essentially, these are skills that involve inquiry and reasoning with the goal of making sense of mathematics.

Research and experience supports this approach to teaching mathematics. The challenge, or the problem, lies in making it happen in classrooms. A great many teachers have never experienced learning mathematics through a hands on, thinking approach; because their own understanding of mathematics is limited, they lack the knowledge and the skills to teach mathematics through an investigative approach that fosters thinking skills. Effective retraining of teachers is an important component of attaining the goals set by the NCTM and others.

Teachers interviewed for this paper who took the Mathematics a Way of Thinking Workshop reported that the workshop experience helped them begin to view mathematics learning and teaching differently. Use of manipulative materials and the safe, supportive environment of the workshop were factors that these teachers reported to be important in facilitating their own thinking and understanding in mathematics, and in helping them to see mathematics anew as accessible and sense-making. In the next section, I shall look at how manipulative materials and a supportive environment support and enhance thinking in mathematics, and how these two factors contribute to a view of mathematics as a sense-making process.

Factors That Enhance Learning to Think in Mathematics

Manipulative Materials. Manipulative materials, sometimes referred to as concrete materials, are "objects that appeal to several senses and can be touched, moved about, rearranged, and otherwise handled" (Kennedy 1986, p. 6). They can be objects collected from the environment, or materials designed to teach specific concepts, such as base ten blocks or fraction pieces. Among researchers, there is an "overriding consensus that manipulatives can help children to understand and use mathematical concepts" (Driscoll 1981, p. 21). Manipulative materials help students to use mathematics concepts by providing mental impressions for

abstract ideas, by providing a way for students to connect the real world to the abstract ideas and symbols of mathematics, and by actively, physically engaging them in solving problems. (Kennedy 1986; California State Department of Education 1985; Kaplan, Yamamoto and Ginsburg 1989; NCTM 1989) There are three ways in which manipulatives are used to develop understanding of concepts. They are:

1. as models for an algorithm
2. as tools for investigating mathematical concepts
3. as aids to solving non-routine problems.

I shall give a brief explanation and example of each of the above.

1. As models for an algorithm. One common use of manipulative materials is as a model for an operation, and for the algorithm for that operation. An example of this is using base ten materials to model the process of carrying and borrowing in addition and subtraction. Such modeling enables the student to see the connection between the numbers in the algorithm and real objects, to develop a mental impression that will eventually enable her or him to make sense of the algorithm alone.

Beattie describes evidence that teaching for understanding through manipulatives improves students' abilities to do computation, because the materials "are the key" to helping students make the connections between the real objects, the language of operations, and the written algorithm (1986, p. 25). Other sources confirm the critical

importance of forging this connection between the real world and the abstract world of mathematics, and the role that manipulatives play in helping students develop mental images or impressions upon which to build abstractions.

When children are taught ...through manipulative materials, the concrete representation provides a vivid mental impression and serves as a referent for later mathematics learning... For children, these connections to the concrete world result in an effective transition to a conceptual understanding of symbolic algorithmic procedures. (Kaplan, Yamamoto and Ginsburg 1989, p.80)

2. Tools for investigation. A second use of manipulatives is as a tool for investigating mathematical concepts which are not represented by an algorithm. Common materials can be used to provide a simulation of such concepts, which can then be used to draw conclusions about real situations. An example of this would be a probability exploration, in which students are given a bag which contains a known number of red and green marbles. The total number of each color, however, is unknown. Students take one marble at a time from the bag, replacing it each time, and tally the number of each color they get. They repeat this several times, then try to make reasonable conjectures about the number of red and green marbles in the bag. Experiences like this can be used as a meaningful basis for understanding probability, and the limitations on the kinds of conclusions that can be drawn from statistical information in real situations.

Kennedy cites the fact that learning theories support the notion that children who have a firm foundation in manipulative experiences are more able to "bridge the gap between the world in which they live and the abstract world of mathematics;" such experiences "help children understand both the meanings of mathematical ideas and the applications of those ideas to real world situations" (1986, p. 6). Using manipulatives to learn mathematics helps students connect their understandings about real objects and their own experiences to mathematical concepts. (California State Department of Education 1985)

3. Aids to problem solving. A third use of manipulative materials is as an aid to solving problems for which there is no applicable algorithm, and for which there is no specific procedure. Materials for such problems can be designed specifically for the problem, such as Tangrams or Pentominoes, or they can be any material the problem solver chooses. The purpose of materials in such problems is to provide concrete representations that can be moved about and used to help think through or visualize a solution. Examples are spatial problem solving, such as Tangram puzzles, and problems such as Ten Men in a Boat, in which ten 'people' must switch places in a 'boat' following specific rules about how and where they can move.

This type of problem solving can be frustrating to attempt without concrete materials. Working out the

solutions with objects helps students to develop strategies for approaching unclear or non-routine problems, and to realize that often a concrete representation can clarify the problem and help to solve it.

Manipulatives as an Aid to Critical Thinking. Manipulative materials are a key to helping elementary students develop critical thinking skills in math. Because they have not developed abstractions with which to reason, children need real things with which to investigate and about which they can make generalizations and conjectures.

Children should be encouraged to justify their solutions, thinking processes, and conjectures ... Manipulatives and other physical models... give them concrete objects to talk about in explaining their thinking. Creating and extending patterns of manipulative materials [for example] and recognizing relationships within patterns require children to apply analytical and spatial reasoning. (NCTM 1989, p. 29)

The Standards stress repeatedly the need to provide manipulative materials as part of "an environment that encourages children to explore, develop, test, discuss, and apply ideas" (p. 17). Multiple examples are given of lessons in which concrete materials are used as the foundation for analysis of pattern, evidence of valid arguments, discovery of relationships, and as a basis for making and justifying conjectures.

It is important to point out that, though manipulative materials are invaluable tools for learning and thinking, such materials on their own will not result in the

development of critical thinking. As Driscoll says, "The teacher stands at the very center of the child's experience with manipulatives, and the teacher's role is critical for the child's success" (1981a, p. 22). It is the teacher's thoughtful use of the materials, and the questions that she or he asks, that result in the growth of critical thinking skills.

Simply using manipulatives in teaching mathematics is not sufficient; teachers must guide children to develop skills in thinking...by very carefully asking questions. More emphasis should be placed on the "how" and "why" questions, and less emphasis on the "what" questions. (Heddens 1986, p. 14)

Manipulative materials or physical models provide access to information that students need to use in their thinking, but it is the clarifying and challenging questions asked by a thoughtful teacher that require students to investigate ideas through those materials, or to justify or validate conclusions that have been drawn from them.

A Supportive, Facilitative Environment. The learning environment is a crucial factor in fostering critical thinking (Costa 1985a; Swartz and Perkins 1989; Glatthorn and Baron 1985), and in developing mathematical understanding (NCTM 1989; Burns 1987; MSEB 1990). The classroom should be an environment which supports a spirit of inquiry and the application of reasoning skills, and in which students feel safe enough to take intellectual and psychological risks. It should be a place in which self-confidence grows, as students

develop their critical thinking skills and mathematical understanding.

It is the teacher who creates this environment, through her or his behaviors towards the students. I shall examine three components of teacher behavior that help to create this environment for critical thinking. They are:

1. organizing the classroom so that students work together
2. developing an atmosphere of trust and respect for others
3. demonstrating that thinking is valued.

I shall then show how these behaviors are related to the NCTM goals for teaching mathematics as a thinking, sense-making process.

1. Organizing the classroom. The teacher must organize the classroom so that students can work with and learn from each other, as well as from the teacher. Group work and discussion is important to the development of critical thinking abilities (Ennis 1987; Costa 1985). Such interaction can be in the form of whole class, one-on-one, or small group discussion or problem solving. The important result is that as students discuss their ideas with each other, question each other, and work to find group solutions to problems, they must clarify their own thinking and support their own reasoning (Hoyle 1985; Paul 1987).

[Children] need to discover opposing points of view in non-threatening situations. They need to put their ideas into words, advance conclusions, and justify them... to discover their own inconsistencies as well as the inconsistencies of others. (Paul 1987, p. 135.

Having students work together in mathematics achieves positive emotional and cognitive ends. Working with others gives a sense of social support that reduces the sense of isolation and anxiety individuals sometimes bring to a mathematics learning situation (Burns 1987; Tobias 1980), and increases the comfort level of students and therefore their willingness to take risks. Also, students in small groups have more opportunity for active participation with materials and in discussions (Davidson 1990; California State Department of Education 1985). Through sharing ideas, helping each other, and finding group answers, learning and thinking are supported.

Explaining reasoning strategies and analyses of problems to classmates often results in new insights and the use of higher-level reasoning strategies...Having to explain one's reasoning allows classmates (and the teacher) to check assumptions, clarify misconceptions, and correct errors in understanding and applying mathematical principles. (Johnson and Johnson 1989, p. 236)

As students work together on mathematical tasks, they must share reasoning strategies, clarify definitions and terms, and argue about the validity of alternative approaches and answers. Often this results in new insights, or clarification of misconceptions about concepts or processes. Mathematics learning and thinking skills both are enhanced.

2. Developing an atmosphere of trust and respect. In order to foster critical thinking, the teacher must develop an atmosphere of trust, and of respect for the ideas of others. To create such an atmosphere, the teacher listens to

and accepts students' ideas, demonstrates that she or he values the thinking process involved in getting an answer as well as the answer itself, and refrains from criticizing or humiliating students (Schoenfeld 1989; Costa 1985).

Costa maintains that this environment, which he calls a "psychologically safe climate" (p. 133), is an essential classroom climate for thinking, and that it is largely created by the "open or extending responses" of the teacher. He defines these as (1) wait time, or giving students time to think about a question or problem and consider answers; (2) accepting responses, in which the teacher accepts a student reply or idea in a non-judgemental way, by nodding, restating, or summarizing the idea; (3) clarifying responses, in which the teacher asks a question that indicates that she or he wants or needs to know more about the student's idea or thinking; and (4) facilitating responses, in which the teacher answers questions or provides information that students need in order to extend their own thinking or solve a problem (1985, pp. 133-135).

As a result of such teacher responses, students learn that the teacher is there to support and facilitate learning and thinking, not to try to dictate it. As a result of a teacher who asks them what they think and why, students have "the powerful experience of having their ideas taken seriously, rather than simply being screened for correspondence to what the teacher wanted" (Duckworth 1987,

p. 131). Through the model of acceptance and respect presented by the teacher, students begin to trust their own thinking and to respect the diversity of thinking approaches among their classmates.

Students learning mathematics in such an atmosphere, in which the teacher values their ideas and their thinking, are likely to be more relaxed, confident in their ability to find answers, and generally to have a positive disposition toward doing mathematics.

Students are more likely to take risks in putting forth their conjectures, strategies, and solutions in an environment in which the teacher respects students' ideas, whether conventional or non-standard, whether valid or invalid. Teachers convey this kind of respect by probing students' thinking, by showing interest in understanding students' approaches and ideas, and by refraining from ridiculing students. (NCTM 1990, p. 13)

This issue of mutual respect for ideas is an important part of the third component of teacher behavior, that of demonstrating that thinking is valued.

3. Demonstrating that thinking is valued. The teacher must demonstrate that thinking is valued in the classroom. The teacher does this by asking questions that show that thinking is expected, allowing time for thinking, and by posing worthwhile tasks.

Questions asked by the teacher indicate that thinking is valued and expected (Costa 1985; Swartz and Perkins 1989). Using the "right sorts of questions" has a major impact on student thinking and reasoning. Such questions as 'Why?'

'What if...?' 'Can you give an example?' 'How do you know?' 'What makes you think so?' 'How did you solve it?' 'What is another way to do it?' encourage students to explore, to validate their thinking, and to explain their reasoning in their own words. Such questions also indicate that the teacher values the ideas and thinking of the students.

By the same token, teachers must encourage students to pose their own questions, and to pursue answers to those questions. Questions posed by students often provide the class as a whole with material for further inquiry. Such student instigated investigations also give students the sense of being in control of their own learning, a powerful motivating force behind involving students in critical thinking.

When higher level thinking, creativity, and problem-solving are the objectives, students must be in a classroom climate where they are in the decision making role, where they determine the correctness or error of an answer based on data they produced and validated (Costa 1985, p. 130).

The kinds of questions asked in a mathematics classroom have a major impact on how students think and reason, and on their understanding of concepts.

Good questions call on students to analyze and synthesize as well as to recall facts... Questions such as the following encourage students to explain, experiment, explore, and suggest strategies: How did you solve the problem? Why did that approach work (or not work)? What is another way to solve that problem? (California State Department of Education 1985, p. 17)

Throughout the Standards, the NCTM consistently gives examples of open-ended questions which invite inquiry and conjecture, as well as of questions that ask students to clarify or extend their thinking and share their strategies and reasoning processes. The primary goal is a classroom "permeated by thought-provoking questions, speculations, investigations, and explorations," so that students "develop persevering and inquiring minds" (1989, p. 23).

Time is another important component of indicating that thinking is valued. Wait time is one part of this - allowing enough time after a question is asked for a thoughtful response. The other part is more a matter of pacing lessons, of allowing long enough blocks of time for students to investigate, discuss, try different approaches, and share solutions. Glatthorn and Baron refer to this as a "more deliberate pace" (1985, p. 52), implying time for deliberation.

The issue of time to investigate and reason about serious tasks and problems indicates that thinking is valued in the mathematics classroom. It is not realistic or fair to students to give lip service to thinking skills, and then to impose a restrictive time frame on tasks which require serious mathematical thinking.

A learning environment that supports problem solving must allow time for students to puzzle, to be stuck, to try alternative approaches, and to confer with one another and with the teacher. Furthermore, for many worthwhile mathematical tasks, tasks that require

reasoning and problem solving, the speed, pace, and quantity of students' work are inappropriate criteria for doing well. (NCTM 1990, p. 14)

Another way that the teacher indicates that thinking is valued in the the mathematics classroom is through the tasks that she or he presents for students to do. Mathematical tasks can provide either practice of mechanical procedures, or the stimulus for students to engage in thinking. If the teacher values the use of critical thinking abilities in mathematics, she or he will pose tasks that "entail problem formulation, problem solving, and mathematical reasoning" (NCTM 1990, p. 24). For example, students can be given the dimensions of various rectangles and asked to find their areas, a mechanical computation task for those who know the formula; or, they can be given a task which involves exploring all the rectangles that can be made with a given perimeter, finding the area of each, and then be invited to draw conclusions about the relationship between perimeter and area. The second task involves students in investigation and in looking for patterns and relationships, as well as in computation.

Students get the message quickly that what we value are the things to which we give time. An environment in which mathematical understanding and critical thinking are the goal will be one in which students work together to investigate ideas and find answers, in which mutual respect for thinking is exhibited between students and between teacher and

students, and in which thinking and sense-making are clearly valued because of the tasks presented and the time given to them.

Mathematics as Sense-making

Mathematics as sense-making is not a means to critical thinking, but rather an end product of teaching mathematics as a thinking process. Teachers who have taken the Mathematics a Way of Thinking workshop have indicated that one result of their workshop experience was that they 'saw' mathematics in a new way - as a process that brought about understanding, or making sense of, mathematics. This was the result of working with manipulative materials in a supportive environment in which thinking was expected and valued.

Most, if not all, of these teachers were products of a traditional elementary mathematics education. I shall define learning mathematics as sense-making, or learning for understanding, by contrasting it with traditional mathematics learning.

Traditional school mathematics. Looking at the traditional mathematics curriculum, we see a long standing preoccupation with what has come to be called 'the basics'. In such a curriculum, the focus is on rote learning, memorization of facts, learning rules and procedures for computation that will get you to the one right answer, and doing endless

routine paper-pencil exercises to make sure skills are retained, or simply to learn to do them faster. It is what Skemp calls "the unorganized collection of rules without reason" which "may be fairly described as a series of insults to the intelligence" (1987, p. 85). Students soon lose any expectation that mathematics be meaningful (Schoenfeld 1989; NCTM 1989; Kaplan, Yamamoto and Ginsburg 1989)), and become passive receivers of these rules and procedures. "Too many students have come to view mathematics as a series of recipes to be memorized...;" however, "mathematical rules, formulas, and procedures are not powerful tools in isolation, and students who are taught them out of context are burdened by a growing list of separate items that have narrow applications" (California State Department of Education 1985, p. 12). Far too many students are more than eager to lay this burden down as soon as they can stop taking mathematics courses; they see no connection between mathematics and their 'real' lives. (NCTM 1989; NRC 1989)

Marilyn Burns describes mathematics lessons from her school days as just such a series of rules and procedures to be memorized:

I remember the myriad of rules and procedures I mastered. I learned to keep my columns neat, carry when adding, borrow when subtracting, add a zero in the second line when multiplying, bring down when dividing, add across the tops but not the bottoms with fractions, multiply across both tops and bottoms when multiplying fractions, turn one fraction upside down when dividing fractions, and always reduce all fractions to lowest terms. (1987, p. 1)

And Ms. Burns was one of the fortunate few - she remembered it all! When she began to teach, she realized that for many children, the rules and steps did not give an orderliness to math as they had for her, but rather were "mysterious methods to be memorized much as one would memorize nonsense rhymes" (p. 2).

In Reshaping School Mathematics, the Mathematical Sciences Education Board (MSEB) presents an example of students giving an answer to a division problem that resulted in an army using "31 remainder 12" buses to transport troops. They had done the computation correctly, but had given no thought to interpreting the answer in terms of whether or not it made sense. "Very little in their experience would suggest the need to interpret the result of a mathematical procedure. For most students, school mathematics is a habit of problem-solving without sense-making" (1990, p. 32).

In the traditional framework of learning mathematics, 'understanding' means being able to remember and follow the rules. If one can do the steps involved in a procedure, and get the right answer, if one can pass the chapter test, then one 'understands' the lesson. There is often the assumption that "students who can perform an arithmetic computation understand the operation and know when to apply it;" however, "test results indicate that students are fairly competent at performing computations, but have difficulty applying their

skills to problem-solving situations" (California State Department of Education 1985, p. 12). Skills learned in isolation that can not be applied to real problems are an indication that true understanding has not taken place.

Making Sense of Mathematics. In order to make sense of something, it helps to know what it is. Mathematics is not just numbers to manipulate according to a set of memorized rules or arbitrary procedures. Yet most who grew up learning it as such are at a loss for how else to define it. One "simple approximation" that serves well from an elementary school perspective is the following:

Mathematics is a science...As a science of patterns, mathematics is a mode of inquiry that reveals fundamental truth about the order of our world. But mathematics is also a form of communication that compliments natural language as a tool for describing the world in which we live. (MSEB 1990, p. 11)

Learning or doing mathematics, then, becomes a search for patterns, for the relationships within them and their relationships to each other, and for ways to describe them and record them. Learning mathematics in this context involves "observations, experiment, discovery, and conjecture...trial and error, hypothesis, and investigation" (MSEB, p. 11). It is an active process that involves interacting with real things and the use of inquiry and reasoning skills, not a passive mastery of concepts and procedures. (NCTM 1990)

Learning mathematics becomes a meaningful, sense-making process when students are engaged in tasks that are interesting and relevant to them, which have a connection to actual experiences or to real objects, and which engage them in actively thinking and talking about their ideas and their reasoning. (Davis 1984; NCTM 1989; Burns 1987) For example, in working through two column addition with base ten materials, students and teacher can use the materials to actually show and discuss how the ones must be regrouped to make tens and so on. These real things can then provide meaning for the symbols in the written algorithm, and the 'rule' for carrying becomes a sensible explanation for what happens with real objects.

Similarly, in the Marilyn Burns lesson, "Fractions With Cookies" (1987, p. 35), Ms. Burns puts the students in a familiar situation - having to share a snack - and presents a problem that is interesting to them. Through her questions and responses, and the cookies, the students solve a problem, and learn some important concepts about fractions.

Burns (1987) lists five guiding principles that she uses in designing mathematics lessons that will build understanding, or sense-making. They are, in paraphrased form:

1. Each lesson must be a problem situation that gives students something interesting to ponder and reason about.
2. Each lesson deals with an important mathematical concept.

3. Children are given the opportunity to talk to the teacher and each other about their ideas, and to put them in writing.
4. Whenever appropriate, physical materials are used to give students a way to verify their thinking.
5. Classes are organized into small groups to maximize students' opportunities to verbalize their thoughts, clarify their ideas, and hear ideas of others.

These guidelines are very similar to strategies suggested by Swartz and Perkins for any lesson in which the objective is learning for understanding. First, eliminate all mechanical problems; then, "emphasize models, images, and metaphors...and connect a new concept to the purposes it serves." (1989, p. 42)

Students see mathematics as sense-making when it is connected to the real world, serves a purpose, can be applied in many contexts, and when they know the reasons behind the concepts because they worked through them, defined them, and created the sense for themselves.

Whether or not mathematics becomes a sense-making process depends upon the teacher. It is the teacher who poses the tasks, provides the materials, organizes the classroom for working with others, asks questions and gives responses to elicit investigation and reasoning processes, and establishes a sense of trust and respect - all of which contribute to learning to think and learning for understanding in mathematics. The teacher is the key. Yet the kind of teaching for thinking and understanding in mathematics described in this chapter is the exception rather

than the rule (Schoenfeld 1989). In the next chapter, I shall look at some of the reasons that elementary teachers approach teaching mathematics as they do, and what effective approaches have been found to help teachers change their approach.

CHAPTER III

CHANGING HOW TEACHERS TEACH MATHEMATICS

The Challenge

If our goal is to make mathematics education a sense-making experience that is meaningful to all students, and which prepares them to deal with the complexities of a technological world, we must not only change what we teach, but how we teach it. Indeed, as Linguist so aptly put it, "It makes little difference what we teach if we do not change how we teach it" (1989, p. 8). The "how" is the key. Almost any content can be taught in such a way that students learn to think. (Swartz and Parkins 1989) The challenge in changing mathematics education is not to change the 'curriculum', or to issue new textbooks, or to provide conclusive evidence from research that supports the need to learn to think and problem solve. It is to change what teachers do, day to day, in their own classrooms.

Textbooks cannot do it. Principals cannot do it. Directives from state authorities cannot do it. Only the people with whom the students come into contact every day can do it. Though many people have vital roles to play, only teachers can finally accomplish the reform agenda (Carnegie Forum on Education and the Economy 1986, p. 26).

There is a serious gap between the ideals of professional practice recommended in the NCTM Standards and the reality of what happens in mathematics classrooms. The challenge that lies at the heart of changing mathematics

education lies in changing what individual classroom teachers do (NCTM 1991). In this chapter, I shall examine some of the reasons that teachers teach mathematics as they do, the new role that has been outlined for teachers by the NCTM, and some effective means of beginning the process of changing teacher behaviors.

Why Teachers Teach as They Do

There are many factors that affect how elementary teachers teach mathematics. Some are related to how little opportunity many teachers have to exercise truly professional responsibilities (NRC 1989), such as: choosing textbooks and other resource materials; the limits, constraints, or demands imposed by administrators; lack of control over the scope of the curriculum; and real or perceived accountability for standardized test results. Rarely are teachers given the freedom or opportunity to use what they know about mathematics and how children learn mathematics to design and implement appropriate curricula.

Other factors that affect how teachers teach mathematics are related to what teachers believe to be true about mathematics and mathematics learning as a result of their own experiences learning mathematics. These include beliefs about what mathematics is, about 'correct' techniques for teaching it, and about what students need to learn about mathematics. These factors also include the anxiety and lack

of confidence about mathematics felt by many elementary teachers.

In this chapter, I shall examine the second set of factors. I believe that teachers teach mathematics the way they learned it, and that the beliefs and attitudes that form the rationale for their teaching are not easily changed.

Teachers' own experiences have a profound impact on their knowledge of, beliefs about, and attitudes towards mathematics, students, and teaching. Teachers' thirteen years as learners of K-12 mathematics provide them with images and models - conscious or unconscious - of what it means to teach and learn mathematics. (NCTM 1991, p. 124)

I shall describe the typical mathematics learning experience of most elementary teachers, and discuss how this experience has affected their beliefs and attitudes, and their approach to teaching mathematics.

How Teachers Learned Mathematics. Most teachers learned mathematics through what Schoenfeld (1987) and Baroody (1987) both call the absorption approach. According to this theory, "mathematical knowledge is essentially a basket of facts and skills" (Baroody 1987, p.7). Learning is controlled from without, by the teacher, who presents information to be learned, and who enforces its learning through reward and punishment.

Marilyn Burns describes a typical lesson from her childhood that describes teaching and learning according to

the absorption model, and it is certainly consistent with my experience.

The teacher usually taught us by explaining and giving chalkboard demonstrations. We students were called upon to respond to questions, then given the chance to try problems at our seats. Sometimes we were sent to the board to do exercises. That was exciting because it was a treat to be allowed to write on the chalkboard, but scary because all the others saw when you made a mistake. Homework was often assigned; then we were on our own or had to call on our parents for help. Thus, we progressed through the math book. (1987, p. 1)

The experience that most elementary teachers have had in mathematics has been presented only in this "authoritarian framework of Moses coming down from Mt. Sinai" (NRC 1989, p. 65). Rules are presented, facts drilled, procedures taught step by step, thou shalts and thou shalt nots spelled out; the chosen people are the ones who can remember it all. The rest generally go through it all again, and in the same way, growing more and more convinced that (a) they are stupid, and (b) math is mysterious, unconnected to reality, and only useful in the theoretical sciences, so who needs it anyway.

Mathematics taught in this way is primarily a passive activity; "teachers prescribe, students transcribe" (NRC 1989, p. 57). Furthermore, it is usually limited to arithmetic operations. Few teachers have had experiences in learning branches of mathematics such as probability, logical reasoning, spatial problem solving, or with non-routine problems that cannot be solved simply by applying an algorithm. They have not experienced the richness of

mathematics as a tool for thinking and daily living. This narrow view, presented from the earliest grades, of mathematics solely as arithmetic, is responsible for "sowing seeds of expectation that dominate students attitudes all the way through college" (NRC 1989, p. 57). This limited expectation or belief about mathematics goes well beyond college; teachers take it back into the classroom when they begin, themselves, to teach.

Results of Learning by Absorption. The learning by absorption model that most elementary teachers experience as students has a powerful influence on the way that they teach mathematics. I shall examine three results of this influence. They are:

1. beliefs about mathematics
2. feelings about mathematics
3. beliefs about teaching and learning mathematics.

1. Beliefs about mathematics. As a result of learning through absorption, which "tends to cultivate...blind procedure following over thinking; mechanical behavior over thoughtful monitoring and problem solving" (Baroody 1984, p.74), many teachers have "unreasonable beliefs" or misconceptions about what mathematics is (Baroody 1984; Schoenfeld 1989; NCTM 1989). The emphasis on calculation exercises promotes and supports the belief that all problems must have a correct answer, that there is only one correct way to solve the problem, and that estimates and trial and

error attempts to solve a problem are unacceptable (Baroody 1984). Because mathematics is equated with arithmetic, rather than with thinking and with searching for relationships, the assumption develops that calculation is always the way to solve a problem. As a result, there is a failure to look for shortcuts or patterns, or to use what one knows or a common sense approach to a problem. Tobias gives the example of a woman who admitted that an idea had occurred to her about how to solve a non-routine problem, but she had rejected it because, "I figured if the question was in my head, it had to be wrong" (1980, p.21).

The emphasis on timed tests, flashcards, and short, calculation oriented word problems fosters the belief that the answers to mathematics problems should always be found quickly (Baroody 1987; Schoenfeld 1987b). Schoenfeld encapsulates typical mathematics learning experiences and the beliefs they foster about speed as follows:

Problems were expected to be solved rapidly, and teachers gave you the solutions if you did not produce the answers quickly. Having had that experience over and over again, you might eventually codify it as the following (implicit) rule: When you understand the subject matter, any problem can be solved in five minutes or less. The stronger form of this rule is even worse: If you fail to solve a problem in five minutes, give up. (1987b, p. 37)

The result of the emphasis on memorization of rules and facts, right answers, right method, and speed is the ultimate counterproductive belief: that mathematics is not supposed to make sense, and is not about understanding (Baroody 1987;

Schoenfeld 1989; Lappan and Schram 1989). These beliefs and attitudes have a strong negative effect on mathematical behavior, causing resistance to any approach to mathematics that does not fit the framework of mathematics as a finite, absolute set of facts to be memorized. And it is very difficult to change that framework.

Behaviors and dispositions are very difficult to change. Once children have established a "facts to be memorized" approach to mathematics, their expectations become a very great constraint to change. Task completion - getting the answers - becomes the goal, and thought goes out the window. (Lappan and Schram 1989, p. 21)

Though Lappan and Schram speak of children, and the effect on them of the "facts to be memorized" approach, it is important to remember that elementary teachers were once children in school, for whom this was the model of what learning mathematics was all about. That model and those beliefs became the framework for teaching mathematics that most teachers carried back into the classroom with them.

2. Feelings about mathematics. Teachers also take their negative feelings about mathematics, which are likewise a result of the way in which they learned mathematics, into the classroom with them.

Historically, the teaching of mathematics has taken place in an atmosphere of rigidity and student fear, as the accumulated knowledge of past generations has been transmitted to anxious students in classrooms devoid of active, engaged investigation (Dossey 1989, p. 22).

Many teachers, like many adults in the general population, have feelings of dislike, bafflement, or despair towards mathematics (Skemp 1987). Dossey refers to a "national distaste for mathematics" (1989, p. 22), and Paulos (1988), in his discussion of the causes for innumeracy, points to widespread feelings of distain for mathematics among adults, and ways in which mathematical issues are avoided.

The message of traditional mathematics teaching, that mathematics is about speed and right answers, can foster what Baroody (1984) calls "perfectionist beliefs" about mathematics. When a student cannot live up to these beliefs, or constantly worries about whether or not she or he will be able to do so, the result can be anxiety and lack of confidence. An inability to learn facts and procedures quickly becomes a sign of inferior intelligence. An inability to answer questions quickly or use a procedure efficiently becomes a sign of being "slow". An inability to answer at all is a sign of real stupidity (Baroody 1984, p. 68). It is easy for a series of failures to begin a downward spiral for students in which the belief that they are mathematically incompetent affects their ability to do mathematics. Knowledge is seen as an absolute, which they might not be smart enough to learn.

Most students - adults and children - believe "knowledge" to be an absolute, which some people have caught on to, and which they, if they are smart enough,

will be able to learn from someone who has caught on.
(Duckworth 1987, p. 131)

Tobias (1980) includes time pressure and the emphasis on one right answer as causes of negative feelings about mathematics, but also discusses isolation and humiliation as significant factors. In the traditional classroom, students work alone, occasionally being called upon to answer orally, or sent to the board to do a problem in front of the class. "At the root of self doubt and unease is a fear of making mistakes or appearing stupid in front of others". This fear often results in a mental block, which creates more anxiety (Tobias 1980, p. 22).

In order to cope with anxiety and self doubt, students develop avoidance behaviors, or protective strategies, which include, among others, not responding or not trying, and procrastination - if you don't answer at all, you can't be wrong (Baroody 1984, p. 71). The anxieties caused by early experiences with mathematics linger long after students leave the classroom; stress and anxiety in the face of mathematical tasks are still there when the child becomes an adult, and so are the avoidance behaviors, albeit in other forms, such as letting someone else figure out the tip at a restaurant, or turning down a promotion rather than dealing with quantitative information (Tobias 1980). Paulos (1988) also gives examples of avoidance behaviors of math anxious

adults, such as avoiding careers that require mathematics coursework in college.

Teachers are not exempt from math anxiety and lack of confidence.

Like most members of our society, elementary school teachers do not have extensive mathematics knowledge, and many have anxiety and negative feelings about mathematics. Yet, unlike others, they must teach the subject to children. (Hyde 1989, p. 225)

It is my observation that teachers have developed their own protective strategies to help them avoid feelings of inadequacy while teaching a subject which makes so little sense to them, and about which they so often have personal anxieties. These include: taking the textbook to be the authority on what should be taught and how, and never wavering from the book; perpetuating the myth among your students that you know all the answers; developing one way to explain a procedure that makes sense to you, and insinuating that students who don't understand it are "slow". These coping techniques may be necessary in order to help teachers deal with their own lack of confidence about mathematics; they also perpetuate the cycle of teaching mathematics as rote drill and practice, and of students who dislike and fear mathematics.

3. Beliefs about teaching. In the context of the absorption approach to teaching mathematics, teaching is telling. Teachers talk, lecture, demonstrate, explain;

students listen and imitate. Baroody explains the teacher's role according to absorption theory as

a well defined and straightforward role for the teacher: to transmit information. The teacher must orchestrate all that goes on in the classroom - presentations, demonstrations, assignments, and rewards and punishments - with that goal in mind. (1984, p. 39)

Because teachers who learned through absorption theory equate mathematics with arithmetic and symbol manipulation, they often believe that "their main task is to teach rapid computation," to "emphasize the rote, the algorithmic, and the procedural" (Hyde 1989, p. 225).

Wilson, who works with preservice teachers, finds that the majority of her students believe that learning is something that the teacher provides for the students to absorb. "...most of my students believe that teaching is telling...No matter what students do on their own, teaching ultimately means that students learn something specific that teachers provide" (1990, p. 206). She quotes a comment from a student, written in response to an open ended class activity in which students were left to reach their own conclusions: "It is my opinion that to teach is to impart knowledge or skills...Today you provided an exercise, but no knowledge was imparted and no skills were passed on" (Wilson 1990, p. 206). To this prospective teacher, conclusions she might draw on her own were not valid knowledge; only the teacher could give her that.

Learning through absorption cultivates blind procedure following when one is learning and doing mathematics (Baroody 1984). This translates into blind procedure following when teaching math as well. Teachers who believe mathematics is calculation, the faster the better, and who see their role as a dispenser of knowledge, are likely to use the textbook, page by page, never questioning what is in the book or the way lessons are presented. The reason offered is often, "It's in the book", as if to say, "It's in the Bible". Timed quizzes or practice sheets are common, as are lots of flash card-type drill and practice. 'Problem solving' means the word problems at the end of the chapter, which can be solved quickly by applying an operation. Students work alone, and within the class there is a definite hierarchy of ability levels; the smart kids are the ones who can remember all the rules and facts quickly. Lesson plans are rigid; the teacher sets out to teach a lesson that has a specific content and a specific outcome that children should learn within a specific amount of time.

Commenting on why teachers teach as they do, Davis makes the statement that:

Because mathematics has often been badly taught, many people misunderstood the true nature of the subject; and, in an unhappy circularity, because so many people misunderstand what mathematics is all about, the subject almost invariably continues to be badly taught. (1984, p. 8)

There is a movement, at least on paper, to change that "unhappy circularity". Researchers are in agreement that, "It will no longer do for teachers to teach as they were taught in the paper-and-pencil era" (NCR 1989, p. 63). Goals for students, examined in Chapter 2, include a far broader range of content and skills in mathematics than those that made up traditional mathematics curricula, and seek to nurture a more positive set of attitudes and beliefs. In order to bring this about, the role of the teacher must also change. In the next section, I shall examine this new role.

The New Role for the Teacher of Mathematics

The new goals for mathematics education emphasize inquiry, reasoning, and problem solving, rather than rote learning. They portray students as actively engaged with materials and ideas, the teacher, and other students in order to construct mathematical meaning. The NCTM cites extensive research to support the notion that

learning occurs as students actively assimilate new information and experiences and construct their own meanings. This is a major shift...to learning mathematics as an integrated set of intellectual tools for making sense of mathematical situations. (1991, p.2)

Teaching as telling won't work toward realizing these new goals for students. The content and process goals, including critical thinking goals, so carefully elaborated upon in the NCTM's Professional Standards for Teaching Mathematics, require new forms of classroom organization,

communication patterns, and instructional strategies. In the context of these new goals for students, the teacher is a facilitator or an orchestrator - the one who guides student learning - rather than the one who imparts factual knowledge (NCTM 1991).

I shall briefly review the teacher behaviors discussed in Chapter 2 that enhance learning to think and learning for understanding, and then examine another important element of the teacher's role that is essential to the realization of the goal of teaching for thinking and sense-making in mathematics, one which I call teaching as thinking..

Teacher Behaviors. There are specific instructional behaviors that have been found to have a direct influence on students' learning to think (Costa 1985; Costa and Lowrey 1989; Swartz and Perkins 1989). I examined several of these in Chapter 2. They are:

1. structuring the classroom for working with others;
2. thoughtful use of appropriate manipulatives and models;
3. asking questions that invite speculation, reasoning, explanation, rather than a single right answer;
4. responding to students in a way that invites them to elaborate upon or extend their thinking; and
5. demonstrating that thinking is valued, through worthwhile mathematical tasks and adequate time for investigation and reasoning.

These are all important aspects of teaching for thinking. However, one other important aspect of the teacher's role might be called teaching as thinking. A teacher teaching in this way is involved in ongoing analysis

and evaluation of the teaching-learning process, constantly applying critical thinking skills to that process.

Teaching as Thinking. I will describe what I mean by teaching as thinking by examining two examples. One, taken from the 1991 NCTM Professional Standards for Teaching Mathematics (henceforth referred to as the NCTM Professional Standards), refers to this aspect of the teacher's role as "analysis of teaching and learning". The other is taken from Baroody (1984), who refers to teaching as "a problem solving experience". I shall then show how each of these models parallel critical thinking dispositions and abilities described by Ennis (1987).

Analysis of teaching and learning. The NCTM Professional Standards (1991) state four major roles for teachers. The first three reflect the teacher's role in attaining goals related to classroom environment, selection of meaningful tasks, and managing tools and classroom discourse. The fourth is stated as: "Analyzing student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions" (NCTM 1991, p. 5). It refers to the teacher's role in balancing the first three goals, continuously monitoring students to ensure that sense-making is occurring, and recognizing the need to change the direction of the lesson if it is not.

One of the four assumptions upon which the Professional Standards are based is that "Teaching is a complex practice and, hence, not reducible to recipes and prescriptions" (NCTM 1991, p. 22). Implied in this assumption is the message that teachers cannot teach the skills and content that are the new goals of mathematics education in the traditional authoritarian way, which relies heavily on scripted lessons from a teacher's guide. To teach for thinking, and to help students make sense of mathematics, a teacher cannot ask a given set of questions that will automatically produce thinking or sense making. Each question asked, each response given, must be based on the students' response to the last question, or to the activity in which they are engaged. In other words, as the teacher teaches, he or she must analyze the success of the task, the effect of the environment, and students' responses in order to make ongoing decisions about what to do or say next. Duckworth refers to this aspect of teaching as having the students "try to explain the sense they are making, and, instead of explaining things to students, to try to understand their sense" (1987, p. 123). If we can understand the sense that students are trying to make of things, we can provide more experiences, or ask more appropriate questions, to help them construct that sense more fully for themselves.

Elaboration of this aspect of the teacher's role is given in "Standard 6: Analysis of Teaching and Learning" (NCTM 1991, p. 63), which states that the teacher of mathematics should engage in such ongoing analysis by:

1. gathering information about students to assess what they are learning through observing and listening
2. examining the effects on student learning of the environment, the tasks, and the discussions.

The purpose of these observations and examinations are to:

- a. ensure that every student is learning sound and significant mathematics;
- b. challenge and extend students' ideas;
- c. adapt or change activities while teaching;
- d. make plans, both short- and long-range; and
- e. describe and comment on students learning to students, parents, and administrators (NCTM 1991, p. 63).

Teachers must engage in this kind of ongoing observation and assessment in order to adjust their teaching to fit the needs of their students, often on a moment-to-moment basis. Unlike traditional lesson plans, which often were written out a week in advance, a teacher who is constantly analyzing the feedback from his or her students will often base plans for the next lesson, or next phase of the lesson, on what is happening now.

In the NCTM Professional Standards, a selection of vignettes are provided, each of which describes a teacher analyzing a teaching and learning situation. In one vignette, second grade students have taken a test on addition and subtraction with regrouping, and many "forgot" to regroup in subtraction. At first, the teacher assumes that the students were careless; then he decides to gather more information by observing individual students work some problems, and having them explain what they are doing. Based on his observations of their difficulties with these problems, he has them use manipulative materials, with which they are familiar, to solve the same problems. Using the materials, the students get the correct answers. The teacher conjectures that he did not provide enough concrete experiences and appropriate discussion to help the students create a mental link between the materials and the algorithm. He now can adapt or revise his teaching strategy, and his plans for subsequent lessons, in order to help students think through and understand not only how but why and when to regroup. (NCTM 1991, p. 65)

Analysis is the systematic reflection in which teachers engage as they examine the relationship between what they and their students are doing and what students are learning. It entails the on-going monitoring of classroom life - how well the environment, tasks, and discourse foster the development of students' mathematical literacy and power. (NCTM 1991, p.20)

Teaching as problem solving. Baroody's concept of the role of the teacher as problem solver is very similar to the analysis role described by NCTM. Baroody sees teaching as

essentially a problem solving process that requires great flexibility... In this view, the teacher acts as intermediary - someone who helps external factors and internal factors to mesh. (1984, p. 40).

External factors are the task, the environment, the tools, and the discourse. Internal factors are the prior knowledge that students bring to a situation, and the sense they are trying to make by applying that prior knowledge to the task. Teaching in Baroody's view is "essentially translating mathematics into a form children can understand" (1984, p. 40); this is done by constantly monitoring what is happening as a result of one's teaching, and changing the approach or the activity, or asking a clarifying question, in accordance with students' responses.

Because every situation and child are different, teachers must continuously make "educated guesses" about how to proceed. Furthermore, they need to check or evaluate how effective their decisions have been. (Baroody 1984, p. 40)

By teaching in this way, teachers are essentially "hypothesis-makers and hypothesis-testers" (p.40). Baroody gives an example of a teacher working with a child who has made a series of errors in basic addition and subtraction facts. Rather than simply assuming that the child needs more practice, the teacher engages in the following "practical

problem solving process", or diagnosis, which Baroody likens to the "scientific method" (1984, p.41):

1. recognize the problem (Child couldn't do the work);
2. formulate a hypothesis (Maybe he can't read the numerals);
3. test the hypothesis (Ask child to read numerals 1 to 9. He does fine);
4. draw a conclusion (Reading numerals is not the problem);
- 2A. revise the hypothesis (Maybe he doesn't understand...);
- 3A. test the hypothesis (Give a simple but related task which tells whether 2A is the problem or not).

The teacher continues the process, until she or he can draw a reasonably accurate conclusion about the source of the problem. The teacher can then decide on a means to help the child construct an appropriate understanding of the task, based on this conclusion, but constantly evaluating the effectiveness of that decision. In effect, the problem solving process is cyclical.

In both the NCTM model of analysis and Baroody's model of teaching as problem solving, the teacher is not following a lesson plan in the traditional sense of the word. The goal is not to make sure that students absorb a specific chunk of knowledge that the teacher has to give them; rather, the teacher has a general goal in mind, and remains flexible and

open to messages from the students and what they mean in terms of the students' progress towards that goal.

I shall show how the two models described above are examples of teachers engaged in displaying specific critical thinking dispositions and the critical thinking abilities Ennis (1987) refers to as decision making skills.

Teaching as a Critical Thinking Activity. The ongoing reflective, evaluative process in which teachers engage when they analyze their teaching and the resulting learning, or problem solve to find the source of a student's misunderstanding, are examples of teachers using the set of critical thinking skills that Ennis calls "deciding on an action" (1987, p. 15).

I shall list each of the skills Ennis includes in this category, and show how they correspond to the behaviors of teachers in the two models given above of teaching as thinking.

- a. Define the problem. In this first step, the teacher gathers information by observing, listening, testing, or other means, to determine where students' difficulties lie. What is it they don't understand? What is the manifestation of that misunderstanding? (i.e. computation errors; wild guessing; confusion and frustration) What is the cause? (i.e. the environment; materials not suited to the student's way of thinking; something unclear in directions or discussion.) This first step is the most important..
- b. Select criteria to judge possible solutions. The teacher defines how she or he will know that a teaching approach works. (For example: students will perform computation correctly and with understanding, choose a

plausible approach to a problem, or express understanding through discussion or demonstration.)

- c. Formulate alternative solutions. The teacher makes a list (usually a mental list) of possible approaches or plans she or he could try in order to help students.
- d. Tentatively decide what to do. The teacher chooses one approach to try with the students.
- e. Review, taking into account the total situation, and decide. The teacher double checks his or her decision. (For example: Because of certain environmental factors, the nature of the concept we're studying, and the confusion expressed by the majority of the group, it really makes the most sense to_____.)
- f. Monitor the implementation. The teacher monitors and assesses the results of the chosen approach or activity to determine if it is or isn't effective. If it is not, she or he attempts to define the problem and begins again.

A skillful teacher is constantly and quickly engaging in this process of assessment, decision making, implementation, and reassessment, as she or he integrates the environment, the tasks and materials, and the meaningful discourse going on in the classroom. "Teachers need to analyze continually what they are seeing and hearing and explore alternative interpretations of that information" (NCTM 1991, p. 64). All of these are behaviors of the critical thinker.

The key to change. Changing mathematics education is desirable, if not necessary, in order to prepare students to function well in our technology and information based society. The need for "mathematical power" in all phases of life, and especially in the workplace, has been established.

The means to reach this goal are stated in terms of how curriculum must change, what students must learn, and what teachers must teach and how they must teach it. Discussions of how we might change are often focused on changing children's beliefs about and understanding of mathematics through how teachers teach it. However, the fundamental change that must occur first is a change in teachers' knowledge, beliefs and attitudes about mathematics; only by changing these will we begin to change teaching behaviors that have been built on a lifetime of often inaccurate conclusions and usually negative impressions about mathematics.

There are many factors which influence the process of changing what and how teachers teach, such as school culture, administrative leadership, and other organizational features of school environments. In this thesis, I shall not examine those factors. Rather, I am concerned with the factors that bring about more personal change, such as changes in knowledge, attitudes, and beliefs. In the next section of this chapter, I shall examine these factors and their contribution to bringing about significant change in how individual teachers teach mathematics.

Retraining Teachers: Effective Approaches

What is the issue at the core of changing how teachers teach mathematics? As I established earlier in this chapter, "teachers teach much as they were taught" (NRC 1989, p. 65).

The experiences that mathematics teachers have while learning mathematics have a powerful impact on the education they provide their students. Prospective and practicing teachers spend many hours in mathematics and mathematics education classes, workshops, seminars... Those from whom they are learning are role models who contribute to an evolving vision of what mathematics is and how mathematics is learned. (NCTM 1991, p.127)

Unfortunately, most teachers' own mathematical experiences have left them with a jaded view of mathematics. As discussed earlier, most came away from mathematics classes with the belief that mathematics is arithmetic operations and procedures mechanically applied. This lack of "clear insight" into the true nature of mathematics, combined with negative cultural attitudes towards mathematics, results in teachers who teach what they were taught and as they were taught.

In some respects, teachers are as much victims as those they teach. Where mathematics is concerned, we have a truly remarkable cultural heritage of phobias and anxieties, misconceptions and myths, stumbling blocks and brick walls. Schools have transmitted this heritage - they merely mirror our culture and society. (Hyde 1989, p. 224)

As adults, teachers are still "victims" of a learning process which was often very intimidating, emphasized getting the right answer quickly, and placed very narrow boundaries

on the definition of mathematics. Few experienced mathematics as solving real problems, exploring open-ended questions, or as encompassing areas that went beyond arithmetic. In their classrooms, they perpetuate this cycle. They simply don't know what else to do.

Any change in how teachers teach mathematics can only come about if teachers have a willingness to change, and as a result of learning experiences that change what they know and think about mathematics, as well as how they feel about mathematics.

Methods of helping individual teachers change or improve must address more than their knowledge...Simply stated, what teachers do in the classroom is a function of how they think about mathematics and how they feel about mathematics. (Hyde 1989, p. 226)

Willingness to Change. No change in the approach to teaching mathematics will occur if teachers are unwilling to change, or see no need for change. Experience has taught us

that evidence of change can be found in textbooks, that the need for change can be presented at professional conferences, and yet that change will not take place if the classroom teacher is neither convinced about the need for change nor ready for the new developments. (Sobel 1981, p. 188)

Even when teachers acknowledge a need or express dissatisfaction with the status quo, bringing about real change is a complex process. It means "relinquishing established systems of thought and action" (Hyde 1989, p. 227), and giving up one's autonomy, being "needy" for help and support,

while experimenting with unfamiliar concepts and techniques. It means a period of risk taking and uncertainty.

It is crucial to consider teachers' willingness to confront their own knowledge limitations and their own anxieties about mathematics and its teaching...we need to create structures and processes that create a desire to improve mathematics teaching among elementary school teachers. (Hyde 1989, p. 226)

Such structures and processes must provide the necessary time and support for the sustained effort required for teachers to change their thinking and their feelings about mathematics and the teaching of mathematics.

Changing Knowledge. Studies of effective teacher training - that is, training which helps teachers bring about real innovation in their classrooms - must result in more than just knowing about the innovation; training must actively engage teachers in learning the new content and skills themselves. (Hyde 1989; Lieberman and Miller 1981; NCTM 1991) "We have come to realize that improving mathematics teaching is more complicated than merely offering a teacher additional mathematical or technical knowledge" (Hyde 1989, p. 225). Hyde lists several guidelines for "facilitating a teacher's cognitive development", which include "significant, direct, and active role-taking experiences, which are appropriate to the teacher's state or stage of development, accompanied by careful continuous guided reflection" (1989, p. 226). In other words, teachers must be (1) involved in actively doing what they are learning, (2) the tasks must be

appropriate in terms of the teacher's ability to understand their content and purpose, and (3) there must be time within the learning process itself for teachers to discuss their thoughts and reactions to what they are learning. Lieberman and Miller concur, adding that "small steps toward an improved practice are more important for the classroom teacher than any grand design," and that being actively engaged in learning about the improvement helps teachers "see the connection between what they are trying to do and the effect of those attempts on students" (1981, p. 54). As teachers do themselves what they will ask their students to do, they understand the thinking processes and conceptual understanding their students will gain from the task.

Teacher training that fulfills these guidelines and addresses the NCTM goals must have two components: improving teachers' knowledge of mathematics, and improving teachers' knowledge and skills in teaching mathematics.

Knowledge of mathematics. In order to improve mathematics teaching, we must extend teachers' knowledge of mathematics. Because of their own mathematical experiences, many teachers are simply unfamiliar with the content and application, especially on the elementary level, of many branches of mathematics. They are also unaccustomed to using mathematics as a tool for thinking about and solving a wide range of problems. Teachers cannot begin to teach

mathematics as an investigative and reasoning process, or make students aware of its applicability to many real-life situations, until they learn to use mathematics in that way themselves, and increase their knowledge in content areas.

Teachers need to conceive of mathematics as a system of connected principles and ideas constructed through exploration and investigation. The ability to identify, define, and discuss concepts, structures, and procedures and to develop an understanding of the connections among them and, eventually, appreciate the relationship of mathematics to other fields is critically important. (NCTM 1990, p. 71)

Learning experiences for teachers must engage them in developing a conceptual foundation of mathematics and in making sense of mathematics (NCTM 1991), through engaging them in hands-on experiences with manipulative materials and other models. "Representations are crucial to the development of mathematical thinking, and, through their use, mathematical ideas can be modeled, important relationships identified and clarified, and understandings fostered" (NCTM 1991, p. 128). Such learning experiences are particularly important for teachers, who very often have had no experience with such models, and therefore often have a shaky conceptual base.

As Hyde (1989) observed, such activities must be appropriate to the teacher's stage of development. For example, if a teacher has no conceptual base because of a lack of concrete experiences, activities using such materials to show the development of a concept must begin at a very

basic level, as you would with a child just learning the concept. For example, if one wanted such a teacher to understand why manipulatives are important for developing an understanding of addition with regrouping, it would be important to start with counting and adding games that develop the concept of why our counting and grouping system has the rules that it has. This basic set of rules can then be related to addition, subtraction, multiplication, and division and decimals, providing a foundation for all of them, as well as a connecting thread. We cannot assume that teachers, because they teach a particular operation, have the conceptual framework that will enable them to understand how a material relates to that operation and the concept behind it unless they have seen the development of the concept from the ground up, so to speak.

Teachers must also have time during the learning process to talk with each other and the leader of the group about what they are learning, and their reactions to it. For many teachers, learning mathematics in this way is almost a shock to their systems, because it conflicts so completely with their previously held views. As their beliefs about mathematics change, and their own mathematical thinking "opens up", they need time to ponder these changes, to process them, share them, and internalize them.

Teachers need opportunities to examine their ideas about mathematics, about the nature of mathematics, about what it means to "know" mathematics, and about their own

learning of mathematics...Teachers' reflections on their own learning can help them to monitor and modify their own thinking and performance. (NCTM 1990, p. 72)

Knowledge of teaching mathematics. The NCTM goals for improving mathematics education call not only for changes in the content of school mathematics, but also for significant changes in how mathematics is taught. In order for teachers to begin to meet these goals, effective training must help teachers learn how to: develop an environment of trust and mutual respect; create a climate for thinking, through appropriate questioning of and responding to students; organize a classroom for communicating and working with others; develop methods for demonstrating that thinking is valued; and begin to develop skills in teaching as thinking.

Teacher training must involve more than teachers learning about techniques for teaching for thinking and sense making; they need to be in a learning situation in which the leader models these techniques (Hyde 1989), and in which the teachers experience the effects of these techniques on their own learning and thinking (NCTM 1991). Teachers involved in such training are "participant observers"; that is, participating as learners and observing the teaching and learning process. As such, they can

assimilate strategies and techniques for teaching particular topics and develop beliefs about teacher behaviors and successful classroom practices. Those from whom they are learning are role models who contribute to a growing vision of what it means to teach mathematics successfully. (NCTM 1991, p. 127)

This approach to teaching and learning mathematics is a dramatic change for many teachers. The idea that a question or problem can have more than one answer; that a lesson can end with a question unanswered, or with an answer that generates another question; that the teacher is not responsible for giving answers to students, but rather for helping them learn the means to find those answers; this is a difficult transition for teachers to make who have always seen their role as essentially authoritarian, and mathematics as essentially static. Accepting this new role means giving up a certain kind of power or control that teachers often feel they are 'supposed' to have in order to 'make' the kids learn. By taking on the role of the student with a leader who models teaching for thinking, teachers begin to discover that mathematics is a creative, active process very different from the passive mastery of concepts and procedures that most of them experienced as students.

Instructors working with teachers should emphasize that "to know mathematics is to engage in a quest to understand and communicate, not merely to calculate" (MSEB 1990, p. 12). Experiences that teachers have in such a learning environment form expectations of what mathematics is and what good mathematics teaching is, and provide a model for trying new things in their own classrooms.

Affective Change. The NCTM Professional Standards state that the education of teachers should foster the development of "dispositions" or attitudes toward mathematics so that they develop confidence in their ability to solve problems, communicate ideas, and reason mathematically... flexibility in exploring mathematical ideas, willingness to try alternative methods and to persevere in mathematical tasks, and interest, curiosity, and inventiveness in doing mathematics. (1990, p. 72)

These goals are quite a contrast to the actual feelings that many teachers bring to mathematics - in the contexts of both doing it and teaching it.

Many teachers have either mixed feelings or outright negative feelings towards mathematics. This is largely a result of the experiences they had learning mathematics. Emotions develop in relation to one's sense of competence, "the ability to achieve one's goals by one's own efforts" (Skemp 1987, p. 193). In learning or trying to learn mathematics, many teachers experienced frustration in trying to achieve what they perceived to be the goals for learning mathematics, anxiety because they became unsure of their own abilities and competence, and fearful of failure. Skemp notes that we fear "that which threatens our self image" (1987, p. 191); and that which we fear, we avoid. As a result of their own negative feelings about mathematics, many teachers have a low motivation to do mathematics, and to teach it.

There is much evidence to support the fact that emotions play an important role in learning mathematics (Skemp 1987; Baroody 1984; Hyde 1989; NCTM 1991), and in learning to think (Costa 1985; Swartz and Perkins 1989). Factors that help to bring about positive attitudes and feelings toward mathematics in all learners are: a supportive, accepting atmosphere like that described in Chapter 2, in which ideas and thinking are valued, and learners are actively engaged in constructing meaning and making sense of what they are doing; and social interaction, in the sense of group support and sharing, and mutual help (NRC 1989; Driscoll 1981; Skemp 1987; Hyde 1989).

It is through successfully doing a variety of mathematical tasks in the context of such an environment that one develops positive attitudes and feelings towards it: confidence in one's ability to do mathematics; security, or a sense of control and a willingness to take risks; and even actual pleasure felt "when we newly understand something" (Skemp, 1987). Positive feelings toward mathematics result in increased motivation to do mathematics.

While in the end, motivation must come from within each student, it can only come when the student feels the excitement of learning, experiences his or her own efforts as appreciated, gets some clarity on goals, makes some connection between the work done in mathematics class and those goals, and feels the confidence and freedom to risk attaining them. (Driscoll 1981, p. 63)

In their own learning or relearning of mathematics, teachers must experience this same environment, feel these same successes, and develop the positive attitudes about mathematics that we wish for them to help their students develop.

All students, and especially prospective teachers, should learn mathematics as a process of constructing and interpreting patterns, of discovering strategies for solving problems, and of experiencing the beauty and applications of mathematics. Above all, courses taken by ...teachers must create in these teachers confidence in their own abilities to help students discover richness and excitement in mathematics. (NRC 1989, p. 66)

Developing such confidence makes it possible for teachers to begin to be less rigid in their approach to teaching mathematics. They no longer feel the need to have the control implied by the "one way to one right answer" approach that is characteristic of traditional mathematics teaching, but can begin to "respond constructively to unexpected conjectures that emerge as students follow their own paths in approaching mathematical problems" (NRC 1989, p. 65).

Another factor that is important in helping teachers shed negative feelings toward mathematics is working with colleagues who have similar goals (NCTM 1991; Hyde 1989). Hyde refers to the need for teachers to develop "collegial interaction", a sense of mutual support, collaboration, cooperation, and companionship to counteract feelings of isolation.

This kind of group support is especially important in the improvement of mathematics education, where affective factors are pronounced. Teachers need to realize that their feelings about mathematics are not unique. They need nonevaluative assistance and reassurance from leaders and their peers that they can overcome difficulties and develop more effective teaching strategies. (Hyde 1989, p. 229)

The support of a group working toward the same goal, combined with a variety of mathematics experiences in which teachers can feel successful, can bring about a more positive attitude toward mathematics, even a sense of excitement, which "makes teachers more willing and able to do mathematics and to teach it well" (Hyde 1989, p. 231).

The NCTM Standards for Professional Development state that the education of teachers should foster the development of dispositions toward doing mathematics, and the development of dispositions toward teaching mathematics. The leader or instructor who teaches mathematics for teachers must model positive dispositions toward teaching mathematics, as well as methods and materials. The leader must demonstrate that she or he values the teaching and learning of mathematics, and believes that she or he can do mathematics and teach it well; she or he must model flexibility in planning and implementing instruction, and monitoring and reflecting on her or his teaching, which I defined as teaching as thinking.

Teachers are largely responsible for nurturing the mathematical dispositions of their students - students' attitudes towards mathematics and their tendencies to think and act in positive ways when doing mathematics...Therefore it becomes important that the

experiences teachers have as learners and in their collegial settings should foster positive dispositions toward teaching mathematics. (NCTM 1990, p. 97)

In other words, the learners of today are the teachers of tomorrow; teachers teach as they are taught. It is possible to change how teachers teach mathematics by changing how they learn it. Such "relearning" cannot address only mathematics content and teaching techniques. It must also immerse teachers in a model of what mathematics teaching and learning should be - a complex combination of interesting problems, tasks and investigations that change teachers' beliefs about what mathematics is; successful experiences within a supportive environment that helps to change teachers' attitudes and feelings toward mathematics and what it means to do mathematics; and, a skilled instructor who models teaching for thinking and teaching as thinking. All of these factors combine to provide the willing teacher with a framework with which to begin making important changes in the mathematics learning of his or her students.

Conclusion

Teachers must learn, or relearn, mathematics, in the way that we wish them to teach it. Not by hearing or reading about the theory or practice, but by actually following the same steps and going through the same tasks and processes that would be experienced by children in a classroom, and by interacting with a teacher who uses the teaching skills that

help to make those tasks and processes successful learning experiences. In the context of the NCTM goals, then, teachers need to relearn in a setting in which they:

1. work on open-ended problem-based tasks with a variety of materials;
2. investigate, ponder, puzzle, look for patterns, draw conclusions, test hypotheses as they learn mathematics concepts and applications;
3. work with and communicate with others about the tasks they are doing, and see different ways of thinking about and solving problems;
4. experience a supportive environment in which their ideas are respected by the instructor and their colleagues;
5. begin to develop positive feelings toward doing mathematics.

Through such immersion in a positive mathematical learning environment, teachers learn about mathematics, about how children learn mathematics, about themselves as learners and teachers, and about teaching alternatives that they know are powerful because they have experienced them themselves.

If teachers are to comprehend the nature of mathematics and mathematical thinking, they must experience as learners the kinds of mathematical knowledge and thinking that we expect them to teach...They must develop some level of confidence and competence in doing mathematics. They must believe that they and their students are capable of learning mathematics. They must experience good mathematics teaching as a model of what they themselves might do. Otherwise, we are asking them to teach in a way they have neither seen nor experienced before. (Hyde 1989, p. 225)

In the next chapter, I shall examine the Mathematics a Way of Thinking workshop as an example of a teacher training opportunity which offers such an experience to teachers.

C H A P T E R I V

DESCRIPTION OF THE WORKSHOP

Philosophy, Goals, and Structure

In this chapter I shall describe the philosophy behind the Mathematics a Way of Thinking Workshop, and the goals of the workshop. I shall then give examples of how the activities in the workshop incorporate manipulative materials, and how the instructor creates a supportive, facilitative environment. I shall show how thinking skills are integrated into the workshop activities, and how the instructor models teaching for thinking throughout the workshop. I shall conclude that these factors together enable participants to develop a new view of mathematics as a sense making process.

The Mathematics a Way of Thinking Philosophy. The Mathematics a Way of Thinking Workshop is conducted through the Center for Innovation in Education in Campbell, California. In the workshop brochure, the educational philosophy behind the workshop is described as follows:

Our workshops are based upon the philosophy that both the teacher and the child must be actively involved in the learning process. Using this philosophy, the teacher becomes a facilitator rather than a dispenser of knowledge. The teacher learns to trust that children have already assembled a considerable amount of useful knowledge on their own. The teacher helps to provide ways for children to organize this knowledge and bring it out in a systematic, logical, usable form. Our goal is to enable children to face a variety of mathematical

situations with confidence and success as they solve problems. Learning is limitless for children who can problem solve. (Center for Innovation in Education 1991, p. 3)

This philosophy is consistent with the goals of the NCTM, and with the goals of critical thinking. All three describe a vision of helping students develop the skills and the confidence to think and solve problems for themselves.

Goals of the Workshop. In the course description given for the workshop, the Center's brochure lists three purposes of the workshop. I shall quote the brochure, then elaborate upon each of the stated goals.

The purpose of this course is to provide teachers with an introduction to an inductive method of teaching mathematics to elementary school children, grades three to six. The method employed is activity-centered and relies heavily on the use of manipulative materials to enable the child to recognize the patterns which occur in mathematical situations.

The course provides the teachers with an educational rationale by which they can evaluate the worth of the materials used and the methodologies employed in the inductive method.

The course has as a parallel purpose that of providing teachers with a framework for implementing the use of a child-centered, activity-centered mathematics curriculum in the intermediate classroom. (Center for Innovation in Education, p. 13)

The first purpose or goal of the workshop is to introduce teachers to an inductive method of teaching mathematics. Many activities in the workshop focus on using materials or a problem situation to generate data, which is then organized so that a pattern can be found. The pattern

is then used to make a generalization or formulate a rule, or form a hypothesis or prediction that can be further explored or tested. Some patterns, once found and described, can then form the basis for understanding procedures or algorithms. Because most elementary teachers learned mathematics through a piecemeal, rule-bound approach, it is a revelation to them to find that mathematics abounds with patterns which form the basis of those rules and interconnect the pieces.

The second goal of the workshop is to provide teachers with an educational rationale, or philosophy. The Math a Way of Thinking program is not meant to be an entire mathematics curriculum that teachers plug into place in the classroom to the exclusion of other methods and resources. The workshop presents a specific set of activities with the purpose of conveying an approach to teaching whatever one teaches in mathematics. The approach is built on a belief in active learning, a vision of classrooms in which "children are encouraged to think, explore, discover, and experience" (Center for Innovation in Education 1991, p. 3). The approach presented in the workshop is based on the philosophy that it is through active learning that students build mathematical concepts and meanings. Through their own immersion in this process during the workshop, it is believed that most participants will recognize the value of the approach, and also become more able to evaluate the effectiveness of other materials and methods that they might use.

The third goal of the workshop is to provide teachers with a framework or model for implementing a child centered, activity centered mathematics curriculum. In order for teachers to feel confident enough to implement this approach in their own classrooms, they must experience it themselves. Within the framework provided by the workshop, the participants take the role of the students. They do the activities in the workshop as their students would do them in the classroom, rather than just hearing about them. They experience the value of small group work, and the supportive environment of which that is a part. Within this framework, the instructor models the kinds of questions and responses that facilitate thinking and encourage various strategies and approaches to solving problems. She or he models the use of a variety of materials, the physical management of materials, and ways of teaching concepts from branches of mathematics that most teachers have never included in their curricula. Altogether, the workshop models a framework within which math becomes an enjoyable, challenging, sense-making endeavor, and in which participants begin to trust their own ability to learn to think mathematically.

The goals of the workshop do not include explicit teaching of critical thinking skills, or of methods for infusing them into mathematics lessons. The overall goal is rather to immerse teachers in an alternative to traditional mathematics teaching methods, so that they can discover,

through their own learning, the value of a teaching approach which is hands on and engages students in thinking. This is a key first step in getting teachers to change their own approach toward teaching mathematics.

In the next section of this chapter, I shall give examples of how manipulative materials are used to help participants relearn math concepts in a more meaningful way, and to solve problems.

Manipulative Materials and Active Learning




Place Value and Regrouping. A series of core activities in the workshop rely on concrete materials to explore the concept of place value, and to model the algorithms for addition and subtraction with regrouping. Dried beans and small portion cups are used along with a 'trading board'. The beans are ones, and are placed in the right hand column of the trading board. When the number of beans in this column reaches the grouping number, they are scooped up and placed into a portion cup and the cup is placed in the second column to the left. When the number of cups in this column reaches the grouping number, the cups are all dumped into a larger cup, which is then placed into a third column to the left, called the "supercup". The activities are done in other bases besides base ten, to provide students with a framework for understanding addition and subtraction using the algorithm in base ten.

When students search materials for patterns in grouping of threes, fours, and fives, and then see these same patterns repeating for groupings of ten, they achieve a far greater understanding of borrowing and carrying than is possible from studying base ten in isolation. (Baratta-Lorton 1977, p. 81)

The activities begin on the first day with a game played in base six in which one bean at a time is added to the trading board until a cup is filled. The game continues until five cups and five beans are on the trading board. The question is raised, "What would happen if I added one more bean to the trading board?" The question is left unanswered by the instructor until the next day. This 'plus one' game enables the instructor to establish the rule for trading up. Next the game is played in reverse, starting with five cups and five beans, and subtracting one bean at a time. The instructor models questions that would be used with students, leading the participants to describe what must be done in order to remove one bean from the board when there are no beans in the bean column, but full cups in the cup column. This establishes the rule for trading down - that if you want to subtract more than you have in the beans column, you must first dump a cup into that column, then subtract.

The above games are repeated with a recording step. Each time a bean is added to the board, the total number of beans on the board is recorded on a vertical, two column recording strip. Recording in this way helps participants to realize that the digits "10" derive their value based on what

the "grouping number" is; in base six, "10" means one group of six in a cup, and no beans in the beans column, and would be read "one cup zero beans". (See Figure 1) The other purpose of the recording is to enable participants to find and describe patterns that appear in the sequence of numbers on their recording strips. Repeating the game with recording in several bases provides the data for comparing the patterns that appear in each base, looking for commonalities, and using these patterns to predict what would be found in bases not tried yet, including base ten.

Cups 	Beans 
	0



Cups 	Beans 
0	0
0	1
0	2
0	3
0	4
0	5
1	0
1	1

Figure 1. Base six trading board and recording strip.

These activities provide the foundation for other activities with the beans and cups in other bases done throughout the workshop. On the second day, participants play a game in base six in which two players, each with her or his own trading board, use a die marked zero to five to 'race' to a supercup. This involves participants in practice with adding random numbers to the trading board, so that with each throw of the dice, the player must assess the materials on her or his board and decide whether it is necessary to trade or not. The game is then played in reverse, starting with a supercup on the trading board and throwing the die to 'race' to zero.

This is followed by using base six dice to create addition and subtraction problems. A trading board is made which has three horizontal sections. Two dice are thrown; one tells how many beans to place on the trading board and one tells how many cups to place on the trading board. Using beans and cups, this number is placed in the first section of the trading board. The dice are thrown again, and the second number is placed on the trading board in the second section. (See Figure 2) The two sets of beans and cups are added together by pulling them all down into the third section of the trading board. The bean place is checked to see if beans need to be traded for a cup; then, the cup place is checked to see if cups need to be traded for a supercup. Sometimes trading is needed in one place, sometimes in both, sometimes





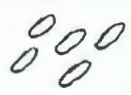
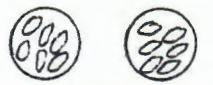

Supercups 	Cups 	Beans 
		
		

Figure 2. Trading board for addition and subtraction.

in neither. Each time, the players must examine the problem and determine what must be done. Similarly, subtraction problems are created by putting one supercup in the first section of the trading board, then rolling two dice for a random number of beans and cups to subtract. Finally, addition and subtraction problems are written on recording sheets in the form of the standard algorithm as students use the beans and cups to solve them; the symbols are a recording of what is happening with real objects.

On the third and fourth days of the workshop, participants work through multiplication and division problems in other bases. It is not recommended that they do this with their students. Rather, the activities serve the purpose of putting teachers into an unfamiliar framework in which they must think about the steps in the standard

multiplication algorithm, and what students must understand about place value in order to understand multiplication and division of larger numbers.

There are several reasons for using lower bases to develop meaning for place value concepts. Working in a lower base with manipulatives requires fewer materials. Students tend to make fewer counting errors, and less time is involved in counting. Trading up or down happens more quickly and more frequently, resulting in more practice with when and how to trade. Above all, however, using bases other than ten takes the participants, and ultimately their students, out of the familiar. Often using the algorithms in base ten has become a mechanical, rote procedure. Using the beans and cups in unfamiliar bases forces the participants to stop and think at each step about what is happening and why, and to evaluate each problem according to the rules established by the patterns. The materials can then become a model for understanding abstract computation in base ten.

Teachers in the workshop are often puzzled at first by why we are working in other bases. Imposing their own rigid framework on their students, they worry that the children will be confused, both by grouping with other numbers, and by the notation when recording in other bases. By the end of the week, however, most teachers report a much better understanding of the number system, and some report that they understand place value for the first time. They are

convinced of the value of the activities with other bases, and see the importance of giving elementary students a strong concrete grounding in place value concepts.

Many teachers really struggle at first to solve problems with manipulatives. They themselves are so unaccustomed to seeing mathematics concepts represented by real objects that they often find it hard to trust the answers they derive with the beans and cups. Their mathematics experience has been so paper-pencil oriented, that it is difficult for some participants to feel that they have an answer at all if it isn't written in a familiar form. I once watched a participant complete a multiplication problem in base five with beans and cups; the answer was before her on her trading board. But until she had translated the problem into base ten, solved it with the standard algorithm, and translated it back into base five, she could not accept that the beans and cups - the real objects - had given her the "right" answer. It takes time for participants to worry less about what the answer is, and to be more concerned with whether they - and their students - understand the process involved in arriving at that answer.

Area of a Triangle. Geometry seems to be one branch of mathematics that gave nearly everyone bad dreams in high school. Elementary school geometry is often limited to identifying two and three dimensional figures and memorizing

some terminology; then in high school the proofs and theorems of plane geometry are a frightening mystery, because they have no connection to any prior experience or real things. The following set of activities from the workshop model the use of concrete materials to explore patterns and relationships in geometry, and to show the connection between patterns and formulas.

Using a geoboard, participants first explore making shapes of given areas. Using various shapes created by participants, the instructor establishes that a square unit can be divided in half with a diagonal line, then demonstrates how this fact can be extended and used to find the area of a right angle triangle. Any right angle triangle can be seen as half of a rectangle; it is easy to find the area of the rectangle, then calculate half its area to find the area of the triangle. The instructor then challenges the participants to use what they know about right triangles to find the area of isoceles and scalene triangles, and finally obtuse triangles, using the geoboard. The area of each type of triangle can be found visually on the geoboard, by finding right triangles and either adding them together or subtracting them away from the original triangle.

After participants become comfortable with finding the areas of the various triangles on their geoboards, they are asked to make a variety of triangles on their geoboards, record each one on geoboard paper, and record the base,

height, and area of each one. The instructor then makes a large chart with three columns, labels the columns base, height, and area, and records data from all the different triangles that have been made by the group. She or he asks the participants if there is a pattern, or a relationship between the base, height and area of every triangle. A member of the group will usually express the pattern with a statement similar to the following: If you multiply the base times the height and then divide by two, you get the area of the triangle. The instructor writes this statement, then points out that the "mathematical shorthand" for expressing the same pattern is $A = 1/2 (BH)$, or $(1/2B)H$, or $B(1/2H)$.

The reaction to this set of activities is often rather dramatic. Though teachers have learned the formula for a triangle, most of them have no idea of how the formula was derived, nor that a formula is a mathematical expression of a pattern, and can be expressed in a variety of ways.

Multiplication of Fractions. Though most adults have memorized the rule for multiplying fractions, and can obtain a correct answer, their answers have no connection to real objects that might enable them to understand why the product of two fractions often gives a smaller fraction. The following activity uses a geoboard and clarification of the meaning of multiplication of fractions to help participants make sense of the algorithm.

The instructor discusses the notion that mathematical symbols, like words, can have more than one interpretation. Whereas with whole numbers, the sentence 3×4 can be visualized as three groups with four objects in each group, most people are at a loss trying to visualize $1/4 \times 1/3$. In the sentence $1/4 \times 1/3$ the multiplication sign should be read "of", so that the sentence says " $1/4$ of $1/3$ ". This means that whatever our whole is, we want to find $1/3$ of it, then find $1/4$ of that third.

Using $1/4 \times 1/3$ and this new interpretation as an example, the instructor makes a rectangle on the geoboard using the denominators of the two fractions as the dimensions of the rectangle. This gives an area that can be easily divided into thirds and fourths. This is the whole that will be used for finding the product of $1/4 \times 1/3$. She or he asks the following series of questions: What is the total number of small squares in the rectangle? [12] What part of the whole is each small square? [$1/12$] (See Figure 3a.) Can we find $1/3$ of the rectangle? [The instructor partitions it off on the geoboard, one row or column of 4 squares] (See Figure 3b.) Can we find $1/4$ of this third? [Yes, it is one square] What part of the whole is the one square? [$1/12$]. (See Figure 3c.)

Following the sequence of questions along with the physical model of each step on the geoboard gives meaning to an algorithm that was previously a meaningless set of rules.

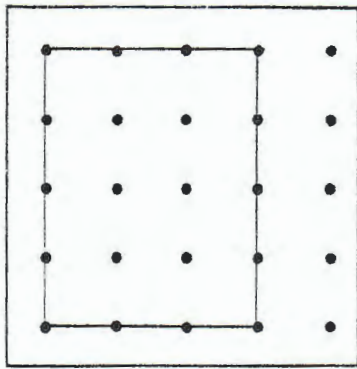


Figure 3a.

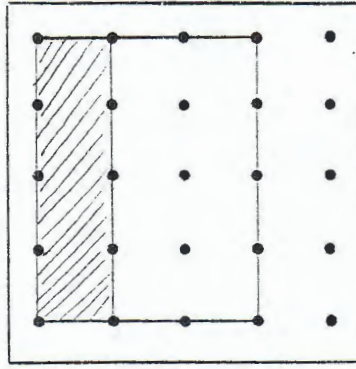


Figure 3b.

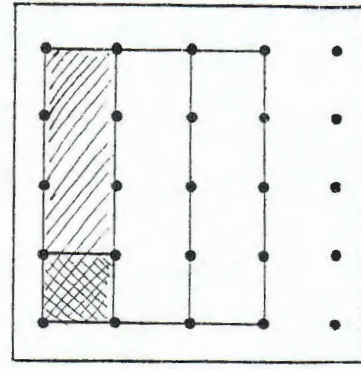


Figure 3c.

Multiplication of fractions on the geoboard.

Probability. In some activities, manipulatives are used to investigate non-algorithmic mathematics concepts through a simulation which can then be used to draw conclusions or generalizations. Exploring probability is an example of this. In the workshop, participants use cardboard "flips" (one inch cardboard squares) and then dice to collect data and look for reasons that explain why some events are more likely to occur than others. Most elementary teachers have little or no experience with teaching probability concepts. This series of lessons is an example of how even very young children can explore the concept through real objects and questions that ask them to think about why things are happening as they are.

Each participant marks one side of a cardboard "flip". They then flip the square repeatedly, keeping track on a two column graph of which side comes up each time. (Cardboard

flips are used instead of coins because they are quiet.) After a time limit, the instructor makes a class graph to represent all the data generated by the class, by graphing which side came up most often for each person. Questions which would be asked of students are modeled by the instructor, such as: "Why do you think the marked side won?" "Do you think the same thing would happen again?" "How could we find out?" The instructor stresses the importance of letting students speculate, rather than telling them answers.

Flipping and recording one flip is repeated several times, until it is clear that the "winning" side cannot be predicted with any accuracy. The instructor then asks what combinations might result if two flips are tossed together. Interestingly, the participants usually agree that there are three possibilities for how the two flips might land: plain-plain, plain-marked, and marked-marked. Participants then work with a partner, combining two flips and graphing the results on a three column graph. Again, the instructor gathers all the data onto a class graph, and models questions which would be asked of students. This is also repeated several times, and the instructor asks the participants to speculate about why the plain-marked combination is always the winner by a wide margin. Often no one can offer any insight into why this happens. The question is left open,

and the instructor offers another exploration which might help participants think about it.

The participants then roll one die, recording which number comes up with each roll. The results of this are also transferred to a class graph, and the activity is repeated several times and discussed in terms of why people think there is no clear pattern to which number will win. It seems clear to most participants that each number on the die has an equal chance of turning up with each roll.

The instructor then asks what sums might occur if two dice are rolled and the numbers added together. Participants then work with a partner, rolling two dice and recording the sum for each roll in a separate column on a graph. When the instructor makes a class graph of the data, the winning sums are clustered in the middle section of the graph. This step is repeated several times, with the same results each time. Finally, a list is made of all the possible combinations for each sum, and it becomes clear that there are more ways to get the sums in the middle range of the graph than on the ends. For example, the sum of two can only be rolled as one and one; seven can be rolled as six and one, one and six, five and two, two and five, four and three, and three and four. It is simply more likely that one will roll the sums in the middle range of the graph because there are more combinations that make those numbers. (See Figure 4)

				3+3	4+3 3+4	4+4				
		2+2	3+2 2+3	4+2 2+4	5+2 2+5	5+3 3+5	5+4 4+5	5+5		
1+1	2+1 1+2	3+1 1+3	4+1 1+4	5+1 1+5	6+1 1+6	6+2 2+6	6+3 3+6	6+4 4+6	6+5 5+6	6+6
2	3	4	5	6	7	8	9	10	11	12

Figure 4. Possible sums with two dice.

At this point, participants are often able to go back and determine why the plain-marked combination came up more often when two flips were tossed. Looking at the flips, it can be seen that there are actually two ways to get the combination "plain-marked" but only one way to get "plain-plain" or "marked-marked".

Manipulatives in Non-Routine Problem Solving. In some activities in the workshop, manipulatives are used as aids for solving problems for which there are no specific procedures. One such problem, given for homework, is called Train Switcheroo. A diagram is given of an oval train track with two sidings, at the top right and top left, and a tunnel at the bottom. Two cars are pictured, one on the right side and one on the left, and an engine is drawn on the right hand siding. The problem is to use the engine to switch the positions of the two cars, adhering to certain rules. Participants find that they very quickly become confused if

they do not use objects to represent the two cars and the engine and then physically move them around on the diagram. Solving the problem is almost impossible without using concrete objects.

In another activity, participants work in groups and use Tangram pieces to explore all the possible ways that they can make specific geometric shapes using different numbers of pieces. Each group sets up a large chart (See Figure 5).


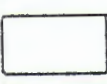
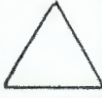




# of Pieces					
1					
2					
3					
4					
5					
6					
7					

Figure 5. Tangram chart.

When a shape is made, it is traced, cut out, and glued to the appropriate space on the chart. If more than one way is found to make a particular shape, both are put in that space. If the group can prove that a given shape cannot be made, they can write "no" in that space. As with the Train Switcheroo, the only way to find "solutions" is by manipulating the objects. In this case, Tangram pieces must be turned, flipped, combined and recombined to discover what shapes can be made from other shapes.

These problems are very different from anything most participants have learned to view as math problems. Few have any strategies to apply to problems that involve visual spatial thinking and a trial and error approach. Many become easily frustrated when they cannot get "right" answers right away. However, throughout the week, as class discussion focuses on ways that individuals or groups approached problems, participants develop a broader definition of what constitutes mathematics, and more acceptance that solutions are not always found quickly and neatly.

Supportive Learning Environment

A supportive learning environment is a key element in the overall effect of the workshop on participants. In this section I shall describe the two components I see as crucial to that environment: (1) working with others in pairs and small groups, and (2) the ways in which the instructor

interacts with the participants in terms of questioning and responding. Another aspect of this second component is the instructor's portrayal of her or himself as a co-learner who doesn't always have the answers, and who enjoys the process of doing mathematics. I shall describe how these components are developed and integrated, and what the results of the environment are for the participants.

Working with Others. Throughout the workshop, participants are seated in groups of four, five, or six, depending on the size of the tables. Starting on the second day, participants are randomly mixed by distributing number cards that correspond to numbers on the tables. The purpose for this is to make sure that all participants have the opportunity to work with a variety of learning styles and approaches, and to hear more points of view, ideas, and strategies than they might if they sat with the same group each day. This seating arrangement also models how students might be mixed for group work, although student groups would not be changed so frequently. Furthermore, it is a non-threatening way to encourage participants to get to know each other, and adds greatly to the 'sense of society' that grows as the week progresses.

During activities, participants work either with the whole group at their table or with a partner from that group. Starting with the first activity, using the beans and

cups, the instructor tells the class to "check your neighbor, have your neighbor check you", to make sure that everyone is together and understanding what is happening and why. This phrase is repeated at times throughout the workshop. Its purpose is to begin to eliminate the idea that sharing ideas and answers is cheating, and to begin making members of the group responsible for one other.

For some activities, participants are asked to come up with a single answer for the group at their table. For example, the class is given a multiplication problem to do with beans and cups in base five. The method for finding the solution is left to each group, but they are asked to come up with a group answer and write it on the small chalkboard provided for each table so that it can be shared with the class as a whole. When all groups are finished, the answers are compared and the instructor invites people from each group to share the method and the thinking they used to get their answer.

In many activities, participants work with one other person from the group. An example is creating addition and subtraction problems with beans and cups. Partners take turns setting up a problem and solving it, explaining to the other person how they are doing the problem and why. Similarly, when finding areas of triangles, participants take turns making a triangle on the geoboard, then explaining to their partner how they determine its area. In order to

explain an answer and the sense behind it, each person must clarify and defend his or her thinking. If one person has difficulty with the concept or explanation, then the partner is there to help them.

Frequently participants work in pairs or small groups to collect data which are then combined with the data from other groups so that the whole class has more information to use in searching for a pattern or drawing a conclusion, as in the probability activities described earlier. In this way, data is collected quickly so that the class can see trends forming and have a basis for discussion and speculation.

Outcomes of Working with Others. Many participants report beginning the class with a moderate to high level of anxiety. They are anxious that they won't understand the material, or won't be able to do the work, or that the teacher will call on them and they won't know the right answer. What if there is a test, and they fail it? Much of this anxiety is born of past experience with traditional math classes.

As the week progresses, anxiety begins to fade and participants begin to achieve a new comfort level with mathematics. The group work and group support eliminate a sense of isolation. It is evident to each participant that she or he is not the only one who experiences occasional confusion or lack of understanding. The fear of being put on

the spot for "the answer" fades, as people become comfortable with the process of arriving at a group answer, and with the idea that there is sometimes more than one answer. Sharing is no longer "cheating", but a means to greater understanding through clarifying one's thinking and listening to other ideas and explanations. Participants find themselves actually enjoying mathematics, as they experience what Skemp called the "exhilaration" that comes of sparking new ideas in each other (1987, p. 88). People become more willing to take risks, and become more able to say "I don't know", because they trust that the group will help them to an understanding.

Role of the Instructor in Creating the Environment. The way in which the instructor interacts with the participants and models a teaching approach which encourages thinking and risk-taking is the key to the effectiveness of the workshop. One aspect of this is the way that the uses of the manipulatives are modeled, so that the purpose of a lesson and the procedures for presenting that lesson are clear. However, the questions the instructor asks and the responses she or he models are the most important factors. I shall give examples of questions and responses the instructor uses, and explain how these contribute to the supportive environment of the workshop. I shall discuss how these types of questions and responses help to open up the participants' minds to a new way of looking at math and math teaching.

Open-ended questions. The instructor asks many questions that have no one right answer. The intent is to generate a variety of possible answers or possible means to get an answer. On the first day, for instance, when participants work in small groups to create their Tangram charts, the question posed by the instructor is, basically, "How many different squares, triangles, rectangles, and trapezoids can your group make using any number of the tangram pieces?" In this activity, it is up to each group to decide when they have found all the possibilities, and all the shapes that cannot be made.

In other activities participants are often asked to look for patterns. The question, "Do you see any patterns?" is always followed by, "Are there any other patterns?" This implies that the first pattern found is not the only, or the right, pattern, and encourages participants to look for other patterns, or to describe those that they might otherwise think inconsequential.

When leading discussions of homework problems, the instructor begins by asking, "Who would like to share one way of finding the solution?" Then all the approaches that have been used are shared and discussed, without any one strategy being labeled as the right one. These kinds of questions are perhaps the most powerful kind to ask people who have always thought of math as a specific set of rules and procedures for getting to a right answer. At the beginning of the week, the

participants are sometimes reluctant to share an approach that they do not think is orthodox, and are waiting for the instructor to tell them which way is the right or best way. By the end of the week, they are delighting in the variety of ways that problems can be approached and solved.

Questions of clarification or challenge. The instructor often asks participants to explain or elaborate upon an idea or solution that they have offered. Answering such questions causes participants to think about and communicate the reasons for their thinking, to clarify steps to a solution, and sometimes to find their own mistakes or unsound reasoning. When participants are sharing approaches to homework assignments, the instructor might ask questions such as, "Can you tell us a little more about how you did that? Do you mean_____? Can you explain what you mean by_____? What made you think of trying that?" Often this type of question is implied in a group assignment. For instance, the instructor will ask participants to come up with a group answer, and to be ready to explain how they got their answer. In this way, mistakes can be found and reasons clarified before the answer is presented, with no one person being held up for scrutiny.

Questions that invite speculation, investigation. Questions of this type ask participants to go a step beyond something they already know, to find out more, or to gather

information that can be used to confirm or alter a deduction or a prediction. An example is "What if..." After discovering why seven is a "lucky" number in the probability activity, participants are asked, "What if you did this with three dice? What would the graph look like? What number would be most likely to come up?" After making a graph that shows the birth order of participants, and speculating about why there are so many people who are first born children, participants speculate about the relationship between occupation and birth order. The question is raised, "What if we graphed this same question with a group of people from a different profession?" In the work with the beans and cups, after participants have worked with one base, the instructor asks, "What if we did this in another base? Would the same patterns and rules apply?" Questions such as these conflict with a previous attitude toward mathematics that saw the solution to each problem as an end in itself. Instead, participants are asked how they can use what they have found out to help them think about or solve a related problem.

Questions that ask for generalizations, conclusions, and reasons. A certain amount of information must be simply given to participants, but as soon as basic groundwork is laid, the instructor begins to ask the participants to reflect on what they've learned so far, and suggest or predict the next step. In the activities with beans and

cups, when the subtraction process is begun, the instructor asks, "What do you think we must do in order to subtract when we have no beans in the beans place?" Such a question is almost always followed by "Why?" When another model for place value is introduced at the end of the week, participants are asked, "Is this less concrete than the beans and cups? Why?" Based on their experience with the beans and cups, participants are able to see why, and to explain why. When exploring patterns for consecutive sums, participants make an organized chart for sums of up to six consecutive numbers and identify patterns that they see. They are then asked, "What pattern would you expect to see for seven numbers? Why?" Questions such as these convey the instructor's expectation that the participants will find their own sense in the work that is being done. It also models a teaching approach in which students are not simply told procedures and answers, but are instead asked questions which lead them to discover procedures, reasons, and answers. Participants discover for themselves how much more powerful a learning experience this is than the traditional teaching as telling approach.

Instructor Responses. Responses refer to the way in which the instructor interacts with participants. These include her or his reactions to ideas, acceptance of various strategies, replies to questions, and attitude toward answers

offered. The general goal of responses is to lead participants to rely less and less on the instructor for answers or approval, and to trust their own thinking and their own ability to determine when an answer makes sense. Instructor responses in the workshop generally fall into the category defined by Costa as "open or extending responses" (1985a, p. 131). These include wait time, accepting responses, clarifying responses, and facilitative responses.

Wait time. Questions in the workshop are followed by time for participants to think about the question, or to use materials to look for an answer. The instructor does not give the answer to a question or a problem. If there seems to be confusion, or no one offers an answer, the instructor asks the same question another way, or suggests an action that might help to clarify the question. (She or he might say, for instance, "What if we did _____, would that work?" The example might be so off-track that it helps by contrast, or a close parallel that helps through its similarity.) Sometimes, questions are left unanswered for the duration of the workshop.

Accepting responses. When answers, strategies, or possible solutions are offered by participants in the workshop, the instructor accepts them without evaluation or judgement. The intent of such acceptance is to encourage the participants to take the risk of sharing their thinking and

ideas, without the fear of being labeled as wrong or off-base. Some activities include brainstorming and data collection, in which all contributions are included on a list made by the instructor. Some may later be scrutinized by the group, and either validated or thrown out, but by then the personal connection is gone. Other examples include the instructor's facilitation of discussions of different ways to solve problems, in which she or he does not validate any one way as correct, but may only respond by saying, "And does that way work, too?" The instructor may paraphrase a statement made by a participant, to make sure that it is clear to the group, but decisions about the validity of such statements are left for the group to decide.

Clarifying responses. If the instructor is not sure of a participant's ideas, or is unsure that it is clear to the rest of the group, she or he may ask the participant to clarify what is meant, by asking questions such as, "Can you give an example? Can you elaborate on that for us? Do you mean...?" Such questions do not imply rightness or wrongness, only the need for more information. This type of a response often has the result of making participants think more clearly about their ideas or solutions, so that they can express it more clearly, or helps them discover an error in their own thinking.

Facilitative responses. The workshop moves at a very fast pace, covering in a week what might be done in a classroom over a span of months. It is important that the instructor be aware of confusion or frustration that may occur, and make it clear that these are not the result of incompetence on the part of the participants, but rather the result of the newness of some things, and the pace of the workshop. Comments such as, "We went through that awfully fast; let's do it again with another problem and see if it makes more sense," or questions such as, "Do you need more time to work on that?" or "Would it help if we gathered more data before we looked for a pattern?" focus the participants on the process, and help dispel fears that they are to blame for their momentary confusion. Facilitative responses also include answering questions that ask for information or clarification which enables participants to solve a problem or understand a process more fully.

An example of questioning and responding. The questioning and responding strategies of the instructor are difficult to separate from one another. Perhaps an example of a discussion from the workshop will help to show how they are interconnected.

Throughout the workshop, daily homework assignments are given which include kinds of problem solving not included in the prior mathematics experience of most participants, such

as visual-spatial thinking and logical reasoning. Participants are encouraged to work on the problems with other people. The next day, discussion of the homework focuses on the different ways that participants went about solving or attempting to solve the problem. The instructor's role in this discussion is to:

1. Ask for clarification (Do you mean...? Would another way to say that be...? Can you give us an example, show us?)

2. Ask for other approaches that people used. (Did anyone do it another way?)

3. Validate all strategies that lead to a solution, rather than focusing on one way that may be the most accepted way. (Clearly, there's more than one way to think about this! That is an interesting way to approach this problem.)

4. If there is no solution, ask questions that help to clarify the problem. (What did you try that didn't work? Why do you think it didn't work? What key thing do you need to know, or what key place do you need to get to, in order to find the solution?)

Sometimes a problem is not solved the day after it is assigned. After participants share ways that they attempted to solve the problem, the instructor does not give the answers, but offers the reassurance that the problem can be solved, and leaves it "out there" for further consideration. Usually, by the end of the week, all homework problems have

been solved, and their solutions shared, but the solutions always come from the participants.

The outcome of such an approach to the homework problems is seen over the course of the week. Many participants are frustrated at first that the instructor will not give answers. They are also sometimes fearful to share their solutions because they assume they did not do them the "right" way. As the week goes on, however, participants become more comfortable with the notion of the acceptability of any strategy that works, and gain faith that a solution will be found by one of the group.

Empathy of Instructor. I believe another important element of the supportive, facilitative environment of the workshop is the degree to which participants see the instructor as a colleague, a co-learner, rather than as one who possesses knowledge, mysteriously gained, that they wish to possess. To be the most effective model possible, I must not be seen in the traditional role the math teacher has always held - that of dispenser of facts and how-to - but rather as a guide along a path I am still finding myself.

In the workshops that I teach, I am careful to share the experiences I had the first time that I took the workshop. I talk about the insights I gained, as well as the confusion and frustration I sometimes felt. I mention the things I have found easy to learn, and the things that have been

difficult for me, including the problems I still can't get the answer to by myself. I am not afraid to say to the participants, "I don't know - I haven't figured that out yet, or learned that yet." For many participants, it is disconcerting at first to have an instructor who doesn't have all the answers. However, part of what I want teachers to gain from the workshop is the insight that that no one knows all the answers; that they can explore concepts with their students, give problems to which they don't know the answers, put students in charge of their own learning through the questions they ask, and students will learn mathematics with enthusiasm.

After the third or fourth homework assignment, I ask how many people have been frustrated or angered by the homework lessons. Many respond tentatively, as though it is an admission of incompetence to say that the problems weren't all a snap. Then I describe how, during my first participation in a workshop, I took the homework assignments home each night in a rage and tore them to little pieces. I was angry that I couldn't do the problems, and also that we were being given homework that wasn't really "math." My anger wasn't helped by the fact that I was staying with someone who went off by herself and produced answers, quickly, to every problem. Then I describe how, as the week went on and I listened to the ways that other people had solved the problems, I realized that I wasn't stupid or math

disabled. I had simply never learned strategies for approaching problems of those kinds. My mathematics education had never gone beyond arithmetic, and I thought that people who could do logic and spatial problems were just "born that way." Discovering that I could learn the thinking needed to do those problems was a major breakthrough for me in my own relearning of mathematics; and I make it clear that that process is on-going, and that I am still learning, too.

Implications for Sense-making. In the Mathematics a Way of Thinking workshop, the active, hands-on learning, and the supportive, facilitative environment provide a framework within which the participants can begin to "see" math as a meaningful, or sense-making endeavor. The materials provide a connection between real things and mathematical abstractions, enabling participants to grasp many concepts with real understanding for the first time. The use of the inductive method, which asks that participants draw their own conclusions and meanings from increasingly complex data and patterns, helps them to begin to believe that there is sense in mathematics, and to begin to trust their own thinking and their own ability to make sense of it. The communication, both among participants and between instructor and participants, relieves anxiety and isolation, and begins the process of erasing previously held attitudes

and beliefs that hindered the development of mathematical thinking.

Among the attitudes that I see developing in participants as a part of this new view of mathematics are the following.

1. Expectations that math will make sense, that it is possible to know not only how, but why.
2. Flexibility in their view of math, in terms of what it includes, how it is done, and who can do it.
3. Confidence that they can do math, though sometimes they may need help (sharing isn't cheating!).
4. Comfort with different styles of thinking, open-ended problems, and unanswered questions.
5. Enjoyment of the process of doing and teaching math.

These attitudes parallel the positive dispositions towards mathematics that the NCTM views as important to mathematics learning and teaching, and also the critical thinking dispositions that apply to mathematics outlined in Chapter 2, taken from Ennis (1987). Those are:

1. Seek reasons. Expecting that mathematics will make sense, and that one can understand the why, encourages one to seek reasons.
2. Look for alternatives. A flexible view of how mathematics is done and awareness of many problem solving approaches encourages one to look for alternative solutions and interpretations.

3. Be open-minded. Comfort with different styles of thinking, and the variety of alternatives that a group can generate, make one open-minded to those alternatives.

4. Seek precision. Expecting that mathematics will make sense, and an enjoyment of the process of seeking precision are parts of the disposition to seek precision, or a correct or acceptable resolution of a problem.

Furthermore, I would include that the workshop begins to develop in many participants a stronger disposition to be sensitive to others. My interpretation of what Ennis (1987) means by this is being aware of, and taking into account, that those around you may be on a different level of understanding than you - either behind you or beyond you - and that this will affect your ability to understand and communicate with each other. Someone who possesses this disposition will actively seek to be understood on the other person's level, or to tailor explanations or demonstrations to their needs. Comfort with different styles of thinking and different approaches to situations is a part of this disposition, and it is important to helping students construct mathematical meaning for themselves.

These dispositions or attitudes are essential to the further development of critical thinking skills. If the workshop accomplishes only the development of these dispositions, it has done a great deal toward establishing

the need, in teachers' minds, to teach for thinking in mathematics.

In the next chapter, I shall examine comments from teachers interviewed in which they discuss reactions to the manipulative materials, the environment of the workshop, and ways that their view of mathematics has changed as a result of their workshop experience.

C H A P T E R V

TEACHERS' REACTIONS TO THE WORKSHOP

In this chapter I shall describe how I went about choosing and interviewing ten teachers who had taken the Math a Way of Thinking workshop, and what emerged from those interviews. I shall present comments from those interviews that show how teachers responded to specific components of the workshop, and how the workshop has affected their overall attitude toward learning and teaching mathematics.

The Process of Collecting Information

The Interviews. I chose ten classroom teachers from grades three through six, who had taken the workshop with me during the last year and a half at various sites. Six of the teachers had been teaching ten to twenty years, two were in their second year of teaching, one was a Chapter I teacher, and one had taught special needs for several years before moving into a regular self-contained classroom. I chose teachers who had indicated in conversations or through written comments that they felt that, as a result of the workshop, they had gained insights in terms of their own understanding of mathematics, and in terms of how to effectively teach mathematics. My interest was in finding out what it was about the workshop that helped bring about such change in those teachers who reported it.

The interviews were generally about an hour long, and I purposefully made them very informal. I asked teachers to talk about why they had taken the workshop, and what aspects of the workshop they felt had had the greatest impact on them. My purpose was to encourage each teacher to talk in a general way about his or her experience in the workshop, and then to look for common threads in what they reported as being important to their sense of growth or change. I sometimes asked a teacher to elaborate on a comment she or he had made, or to explain how a personal insight had transferred in terms of her or his approach to teaching, but I did not go into the interviews with a specific set of questions, or a particular direction in mind for my conversation with the teachers.

What Emerged from the Interviews. Three things clearly emerged as important factors in changing the thinking of those teachers interviewed. These were: use of manipulative materials; a supportive environment for thinking and learning; and the resulting emergence of a new view of mathematics as a thinking, sense-making process. Teachers also were excited by the changes they were beginning to make in their approach to teaching mathematics as a result of their own experiences in the workshop. In the remainder of this chapter, I shall present and discuss comments from teachers as they pertain to the following categories:

1. The effect of using manipulative materials in terms of seeing mathematics as sense-making, and as a means to gaining insights into concepts previously learned as rote procedures.
2. The effect of the positive, supportive environment of the workshop. Teachers reported that the environment enabled them to let go of old anxieties about mathematics and to begin seeing it as an accessible, thinking process.
3. The overall effect of the workshop on teachers' beliefs and attitudes toward mathematics.
4. The effects of the workshop on how the teachers approach teaching mathematics.

As I present their comments, I shall identify the teachers by using a pair of initials as follows: LG, DC, LC, BP, EG, JC, SC, RS, EO, and LK. This will enable the reader to examine the comments of specific teachers if she or he so desires.

Effect on Teachers of Using Manipulative Materials

The teachers interviewed all learned mathematics through the traditional approach, in which concepts were presented abstractly, and rules, procedures, and facts to be memorized made up the daily lesson. Some had dabbled a bit with using manipulatives with their students, but for all of them it was their first experience learning mathematics concepts through

concrete materials and models. Perhaps because elementary teachers struggle so hard to teach their students addition and subtraction with regrouping, the place value activities with beans and cups have a powerful impact on teachers, as do the geometry lessons with tangram pieces and geoboards. Teachers are struck by how easily these "difficult" concepts can be grasped when they can actually see what is happening and why. As many teachers said, it simply "makes sense".

New Insights into Old Concepts. For some teachers, the materials actually enabled them to understand math concepts that they had memorized as procedures or formulas as students themselves, but never understood in terms of why or how they worked. In response to the general question, 'What about the workshop made an impact on you?', LC said:

Getting the understanding through the manipulatives. From the way that I learned, I was presented with how to do the computation and then later learned to understand what it was all about. This to me was giving the understanding first and then showing how to put it on paper, and it just made so much more sense coming from that direction. It helped me to understand things that I maybe didn't understand.

This sense of finally really understanding concepts is reflected in a comment by LK, who said, "I guess one of the biggest impacts of the workshop was doing the activities with the materials." She goes on to talk about the place value activities specifically, concluding, "it finally made sense to me to do it myself, in terms of why we're doing what we do in place value."

EG, commenting on which manipulatives in the workshop made the greatest impression on her, said, "The beans and cups. And area of triangles on the geoboard. I always knew the formulas, but I never knew how they were derived. It was a real insight to me." LC reported a similar insight from another geoboard lesson: "The one [activity] that sticks out most in my mind is multiplying fractions on the geoboard. It just made so much sense, and I couldn't wait to come home and show everyone how to do it."

This new insight into old learning, and the resulting excitement over making sense of something for the first time, is conveyed in BP's reaction to concrete work in lower bases:

My husband [a scientist] would say, 'We did such and such in base whatever', and I would just let that go right over my head, because I didn't like bases, I was confused by them in school, and I didn't want to know about it. But it's so simple once you do it with the materials, concretely. It made perfect sense!

Insights into Implications for Teaching Children. As a result of their own insights teachers recognize the power of the hands on work in terms of their students' understanding. Because it made sense to them, they see how it will make sense to their students. As RS said:

Doing the activities with the manipulatives has really made me see how I would function as a student in the math program, and how I'd do so much better going from the concrete to the abstract; just going through the whole process made so much more sense to me.

EO also commented on the value of "being the student" in the workshop and actually going through the process of doing each activity:

I discovered a long time ago that I'm a definite motor-type, kinesthetic learner, so having to sit there and do all the stuff really got me understanding how you'd do it with kids...the cups and beans in another base would never have made sense to me if I hadn't had to do it.

Teachers who had already implemented some of the workshop activities in their classrooms, using the hands on materials and their own new insights, reported that the process of learning place value, for example, was much smoother, and made sense to the students, because they had real objects to connect to the process they were learning. SC comments on this: "The way the materials are used - by giving them something to grasp, to look at - it's not just an abstract number on a piece of paper. It has a connection. It means something to them."

This is also BP's observation, after working on place value with her students: "I see that when they're doing addition and subtraction with regrouping with the beans and cups, they're really looking at the process of what they're doing, they're not just manipulating the base ten numbers."

EG talks about how the experiences with beans and cups gave her students a model or mental image that helped them understand the pencil-paper algorithm for base ten: "One of the kids said when we got into regrouping in subtraction [with pencil-paper], 'Oh, that's what we do with the cups and

the beans - that's dumping.' I've always told them, but now they were seeing it." Implied in this last sentence is also EG's new understanding that simply telling students a procedure does not convey the concept behind the procedure.

LK similarly reports how her students were able to make a connection between lower bases and base ten: "I know my kids last year did not understand place value. When I brought in the base ten blocks this year [after doing beans and cups activities in lower bases] I wasn't sure how they were going to do, and they caught on so quickly."

Teachers repeatedly use the phrase, "it made so much sense", or "it just made sense", to express their reactions to the manipulative materials. As RS comments, this new "sense" that they found for themselves leads to an improved ability to provide more meaningful learning for their students.

When you did the algorithm for division, after doing multiplication with arrays, using the division sign and showing how it comes from [can be thought of as the corner of] the array, that just made so much sense! Everyone was saying, 'Oh, that just makes so much more sense for kids! - for adults! We should be using more of that type of approach to teaching, so kids have an understanding of what they're doing. And I understand it, too, and it's easier for me to explain it.

The fact that teachers must understand a concept first themselves, and feel confident in their own understanding of it, before they can teach it effectively, is reflected in many teachers' comments. Their experiences in the workshop w

with hands on materials, and the deeper understanding that it gave them, enabled many teachers to finally try new approaches to teaching. Speaking of the things in the workshop that made an impact on her teaching, DC said:

All the different things we did that I never did before, like the tangrams and geoboards. I just stayed clear of that stuff, because I just never understood it or felt comfortable with it. I could never have taught it because I didn't understand it. I was forced to do it [in the workshop] and saw that it was possible to do it with my class.

For many teachers, the new insights that they have as a result of working through concepts with manipulatives give them a new confidence about doing mathematics. The materials make possible and accessible what always seemed scary or inaccessible. This comes across when BP says:

As a child I always felt threatened by math. I felt insecure about it. It was always a block to me. After the workshop - I can see where the manipulatives make a big difference in your ability to do math and feel confident about it. I felt much better about math when I finished the workshop, because it just made sense to me, I could see it. It made so much more sense than just doing it on paper... it did a lot for my own self-confidence, because for instance bases were like something from another planet, and they're so simple! But you have to do it - that's why it made sense.

Effect of a Supportive Environment

Several of the teachers interviewed, like BP, indicated by their remarks that they disliked math, were uncomfortable with it, or would describe themselves as having some degree of math anxiety. The number of teachers in general who refer

to the workshop atmosphere as "non-threatening" has led me to realize that most teachers come into a math learning situation expecting it to be threatening or stressful. Fear of failure, of not having the right answer or of not doing it the "right" way still are major concerns for teachers whose mathematics learning experience has been limited to the traditional method described in Chapter 3.

The environment of the workshop begins to free teachers from negative attitudes towards mathematics and towards themselves as doers of mathematics, by putting them into a learning situation in which they feel secure and successful. Isolation is eliminated through small group work, lessening anxiety about one's ability to figure things out alone. A sense of success is nurtured through open-ended tasks that can be solved in a variety of ways, and an instructor who clearly values each participant's contributions and thinking. In such an environment, teachers begin to believe that they are able to do mathematics, and to take risks in terms of trusting their own thinking, and by daring to step outside the traditional and narrow mathematics boundaries that they had known.

Shedding Anxiety. Several teachers allude to the breakdown of a fearful or anxious attitude toward math as a result of the workshop experience. For EG, there was a new view of what she had been raised to think of as "cheating". She reported that one aspect of the workshop that had an impact on

her was "...sharing answers. Feeling free enough to not do something and watch someone do it; and to know, it's not copying - it's learning!"

JC also shed the traditional view that mathematics should be done in isolation: "I think you introduced it - the sharing with the rest of the group, and I'd not thought of that, because here we come from the traditional school: 'Everyone be quiet and learn for yourself and think for yourself'."

DC likened the environment to that of the writing process, in which the students coach each other and the teacher guides, rather than acting as an authoritarian dispenser of factual knowledge. "You were teaching us, but when it came down to solving the problems you were just one of us. 'Oh - well, is that the answer? Is there another way to do it?' And there was no one right way, and we were all in it together, and that's something I'd never had in my math curriculum before."

The sense of being "all in it together" is echoed by LK, whose comments also put the environment of the workshop in stark contrast to the mathematics learning she had known in the past, where right answers, quick answers, and fear of embarrassment were the legacy:

Actually, when the course began I was apprehensive about the group work, just from math anxiety, nervous that I wouldn't be able to contribute. What was really helpful was seeing that everyone was in the same boat, and knowing that what was important wasn't the answer, or

that I didn't have to have an answer in a certain amount of time. And knowing that you weren't going to be calling on me. That I could work with the group - that it wasn't just me. By the third day I wasn't worried at all. The emphasis was on seeing how someone else derived an answer, and how it could be different.

Security through Working with Others. Small group work was a very important part of building a sense of security about mathematics for many of the teachers interviewed. BP was one of the teachers who had been very "math anxious", and for whom the group work provided important support. "The cooperative learning was great. I really see the value of working together. Because you don't feel alone. You don't feel isolated, that you have to come up with something alone." She goes on to speak again about how much she liked ... "communicating with others, being able to work as a contributing member of a team rather than in isolation." Her past fears of inadequacy or failure had been the result of isolation. In the workshop, she was not alone in her search for answers; the cooperative investigations and search for solutions made it possible for her to contribute to an overall group success, so that

The things that made me not feel that I'd failed was to be able to work with other people, and develop new strategies, and be a member of a team...it was, like, try the problem, if you can't do it that's OK; sharing strategies for a problem made it non-threatening; it was low-key. It's like wading out into the water and you're not afraid of drowning.

Several teachers reported feelings similar to those of BP, indicating that being able to work and share with a group

or a partner helped them shed old anxieties and enhanced their learning. EO's comments were typical:

It was like I was learning while I was playing. Being able to work with others and talk with others while I was doing it... it was very non-threatening. When I get confused I like to turn to somebody else and see what they're doing. I learned a lot doing it that way too, because of the support of the group...and I think it made it much easier to learn.

In a similar vein, JC's remarks show how the group work made her secure enough to say "I don't know", or "I'm lost", and to feel OK about not having the "right" answer: "The other group members could help you out when you get kind of messed up. You could say, 'How did you do that?' Everybody was very willing to share."

Insights into Teaching. Working in such a positive environment themselves, and experiencing the powerful effect it has had on their own learning, makes teachers aware of the value of establishing such an environment in their own classrooms. Savoring their own sense of success in the workshop, and in many cases, a new sense of enjoyment of mathematics, teachers want the same experience for their students, and now know how to begin to make it happen. BP, for example, reflects in this comment what she has come to realize about the traditional math teaching approach: "I think that's why kids get discouraged in math - when they can't do it, and they're by themselves, and they feel like

they must be a failure. It's made me more aware of how the kids feel. I really felt put in that position."

JC, a Chapter I teacher who works with small groups of students at a time, has been gratified by the way students have responded to working with each other. "I find now in the classroom where I have the small groups, they're always there to help each other. If someone doesn't have all of the information, one of the rules is to ask the other people in your group, not the teacher. And they do seem to learn more that way."

RS also comments on the change in her classroom as a result of group work, as well as her own willingness to direct rather than dictate:

It's a nicer attitude to go with in the class, and they seem to be a lot happier about doing math. It's contributed to a more positive atmosphere...It's just so much more relaxed. I feel I'm learning with them while they're doing it. [Speaking of new approaches and activities she's trying with her class] We're all trying it together. I haven't been able to solve some of the problems myself, but I find with their insights that I'm much better at solving them, too. So we're kind of going through the process together.

Overall Effect on Beliefs and Attitudes about Mathematics

As a result of the hands on work with interesting tasks from a variety of mathematics strands, and an environment that is supportive and encourages reasoning and flexible thinking, teachers report sometimes dramatic changes in their ideas and beliefs about what mathematics is, and about

teaching and learning mathematics. Many teachers' comments reflect a broader view of the scope of mathematics, and a new sense of confidence about doing and teaching mathematics.

Changed Beliefs About Mathematics. Several teachers, like LK, expressed a drastic change in their view of what mathematics includes. The activities in the workshop enabled them to see the importance of skills and concepts other than traditional arithmetic operations, and to begin discovering mathematics as a thinking and sense-making subject.

My whole belief system about math changed in terms of what it is and what it includes. Just learning about all the strands...that math isn't just adding, subtracting, multiplying, and dividing. That there are so many other things involved in it...What the workshop has changed is my thinking. It's totally changed my outlook on math and the importance of math in the curriculum.

In this comment, RS also talks about breaking away from her traditional narrow view of mathematics as computation, and about the fact that this has enabled her to broaden and enrich the mathematics experience she offers her students:

Before, I was into computation - math was computation. Now I'm really looking into what the theory is, and how to explain that to the children so it's understandable. And how to make it much more meaningful for them, so they're not just doing the lesson for the day and then forgetting about it. I'm trying to make them see things as a whole.

LC contrasts her past mathematics experiences, in which math was learned as facts and procedures, with the insight she gained as a result of the workshop. Mathematics now is something based on reason and logic, which she can expect to

understand: "The way we learned math was rote, and we didn't understand what we were doing, we just did it. After the workshop, I discovered that there was some reasoning behind it, that there was logic to what you were doing, other than because the teacher told me to do it this way."

SC also shares the insight she has gained as a result of relearning concepts through a thinking, active learning approach:

We were always taught rote. We were always taught rules for the way you did something, and you followed the rules. But we were never taught why... all of a sudden as you were going through these things [rote rule following] you might begin to see why, but you were never told why, you were never shown why, you were never led through the why first. This way, it's like the light at the beginning rather than at the end of the tunnel. It's an approach to a better and broader understanding, rather than a very narrow view of how to do something.

RS expresses the experience of many teachers when she says, "I was always good at math, I was always able to solve the problems, but I was never any good at explaining why; the workshop has changed my understanding, in terms of why we do what we do."

New Flexibility. As a result of their traditional mathematics learning experiences, most teachers begin the workshop with the rigid view that there is one "right" way to get an answer to a problem. In contrast to this, many comments from those interviewed convey a new flexibility in their attitudes toward how mathematics is done. A more

flexible attitude is cultivated by the instructor, who always asks for alternative ways of thinking about or doing a problem, but without labeling one strategy as the better one. As a result of sharing ideas and strategies, and listening to the thinking of others, participants are more open minded about how answers can be found, more willing to look for alternatives, and more comfortable with unanswered questions.

EG comments on the effect of this open ended kind of problem solving on her own attitude toward doing mathematics when she responded to the question, "What aspect of the workshop had the greatest impact on you?"

The questions! I loved the way you asked open-ended questions. I always felt that things had to be boxed in when I was through; this is the end of the lesson, no loose ends. And I don't any more; I leave a lot of loose ends, and they [her students] love it. And they might come back a week later with an answer to something, and I think, 'Ah! It does work!'

LG reports that as a result of the questions asked in the workshop, she always asks for other possible ways to get a solution to a problem, and doesn't declare one way to be the right way. "As a result, a lot of the kids hear ways to solve a problem that would never have occurred to them, and that make sense to them."

BP contrasts this new sense of having "options" with the traditional view she previously held. Reflecting on the emphasis placed in the workshop on finding multiple strategies for problems, she said: "I think years ago we used to think

math was very concrete [meaning cast in stone, rigid], that there was only one right way to do things, and there really isn't."

For LK, the workshop provided insights into various ways to think about non-routine types of problems that had never been included in her math experience. "One of the things that made an impact on me was working through the homework problems, trying to solve them on my own, then listening to how people did them, and seeing the importance of having different strategies to get an answer."

EO similarly found that the workshop problems helped open up her own thinking in terms of how to approach a variety of problems, instead of assuming that someone had to show her the right way to solve each one.

Probably what the course showed me more than anything else was how to begin to think those ways...the workshop has gotten me thinking better, and I at least know how to approach them, have my foot in the door for how to begin working with them.

This more flexible attitude opens the way for a richer learning experience, both for the teachers in the workshop and, as RS reports, for their students:

I've changed my attitude to: there's not just one way of doing things. Now - people come up with other options or other answers, and I say 'yeah! It makes sense to me; put it on the board and we'll talk about it and see what those other options are.' And I find that a great learning experience, for both the child who's making the presentation, and for myself, to see how other people operate and think...I'm much more open than I've ever been before to looking for alternative ways to solve problems.

New Confidence. Many teachers who entered the workshop with "math anxiety" left with a new sense of confidence in their ability to do math and to begin teaching math more effectively. In some cases, the workshop experience made participants realize that often their unease with math was not a result of lack of ability, but more the result of the rigid and narrow range of their own past mathematics learning. Like LK, they began to realize that anyone can learn to do mathematics and to think mathematically; it's not an innate ability: "It really changed my idea of someone being mathematically inclined. Now I think it has more to do with experience and exposure." That realization boosted her own confidence and interest in mathematics, and she sees this as being contagious:

The course has made me much more confident in math, and made it much more exciting to me. It's a big one! Before, I never really understood what I was doing... Now I'm much more comfortable with math; I'm looking for patterns, which I never did before. So then I bring that attitude back to the class, and try to do the same thing for them, and when I see it's working it really validates it [the approach].

RS also comments on her new confidence in and enjoyment of mathematics, and the effect that has on her teaching, when she says:

I think the workshop has definitely changed my attitude toward math. One, I like it more myself, I find it more fun to teach, and the kids are more excited about it, too. That's because I'm not just interested in addition, subtraction, multiplication, division, whole numbers, fractions and decimals any more. I'm really into making it a part of their lives.

Perhaps one of the most dramatic changes in attitude and confidence are reported by BP, who confessed to high "math anxiety" at the beginning of the workshop. Her school was reorganized into team teaching units the year she took the workshop, and she volunteered to teach math to all four fourth grades.

It [the workshop experience] really made me feel better. I mean, I'm teaching all the math for fourth grade this year, which is a big step for me. I was excited by the workshop, and wanted to see how it worked with the kids. It's made me feel more confident about myself in math -being able to teach it, and feeling as though I'm teaching it successfully. I feel most of the children enjoy coming to math; I feel good about what I've done.

LG also comments on her feeling of being a better mathematics teacher than she was in the past. Because of her own deeper understanding of concepts, and her ability to engage her students in thinking more about what they are doing, she says: "I see myself as a good math teacher now, where before I was just adequate."

Effect on Approach to Teaching

In the interviews, teachers talked a great deal about how they had begun to change their approach to teaching mathematics. As a result of the activities and teaching style modeled in the workshop, and the changes in their own beliefs and attitudes about mathematics, many teachers reported that they were beginning to pull away from the rigid traditional teaching role which was all they had ever known.

They were beginning to incorporate more of the behaviors that facilitate thinking and build a supportive learning environment, and to use their new understanding about how to make sense of mathematics to make judgements about the value of lessons and materials they had used in the past.

Questioning. All teachers interviewed described themselves as asking what BP refers to as "more thinking kinds of questions... that focus more on the thinking and how they did things." Teachers comments show their concern now for process, thinking, and strategies, in contrast to a past concern for facts and correct procedures. EG refers to this change in the focus of her questions when she says:

Instead of asking a specific question that requires a specific answer, once they've come up with their answer I ask them to explain why they said it, and is there another way of doing it. I validate that their thinking is good or logical, though it may not be what I'm thinking, so they don't feel as though they've done something wrong... I have de-emphasized learning of the 'facts' in favor of spending more time on other things... and more time on discussion of different ways to get answers.

RS also contrasts the kinds of questions she asks her present students to those she asked before she took the workshop:

I think I tended to just ask questions and then not follow up on them. I think now I stay in there and I'm looking for more information, about how they're solving their problems. I kind of hang in there longer than I did before to see if we can clarify things. And sometimes I just let there be quiet and see who comes up with some good ideas. I tend not to speak out as much and let the kids resolve some of the problems themselves. And they do!

LK describes herself as also asking questions that help her students to resolve problems for themselves, so that they begin to rely on their own thinking: "The questions I'm asking are to try to initiate, their thinking, in coming up with the answers themselves...And actually, now, they're asking the questions that initiate those responses."

SC gives examples of questions that she asks her students in order to get them to justify, clarify, or elaborate on their thinking:

I'm asking different kinds of questions... asking if they can carry out the same thing in a different way, and re-explain it, showing that they understand it. Asking questions such as 'How did you do this one? Why did you do it this way?' Challenging them to explain how they thought through something. What materials can you gather to prove what you think is correct? Now can you go ahead and prove it?

EG also describes herself as asking different kinds of questions as a result of the questioning that was modeled in the workshop, and how she sees the value of not trying to put closure on every lesson:

I ask things like, 'Why did it work?' or 'Why didn't it work?' or 'How might you do it differently the next time?' I see the value of analyzing what you've done. I always tended to say, 'OK! We've finished that!' It's important to leave those five minutes at the end to talk about what you've done, because it gives them something to build on...You did it to us - it always gave us something else to think about, instead of, OK, this is over. It's, well, perhaps the next time we'll do it this way, or we'll try that, or what if we did that? So it isn't ever really over.

Teachers show through these kinds of questions that they value the thinking of their students, and that mathematics is not simply a matter of memorizing facts.

Goals for Students. Changing the kinds of questions that they ask their students reflects a change in teachers' expectations for student learning. Their comments show that they have less concern for factual knowledge, and more concern that students learn to think mathematically and develop positive attitudes toward mathematics.

In the following remark, RS describes how her focus has changed from the traditional, one right answer approach, to teaching her students to look at mathematics as a thinking process:

The workshop made me want to restructure a lot of what I did in the classroom...I've come to the conclusion that if we are just looking for the answer, and never concerned with the process, that kids lose out, because they could be just slightly off at the end, but they could certainly have the process. We're too concerned with the end product, and not enough with what we do to get there...Now I'm always looking for activities to bring into my math lesson to make it more meaningful to them, connect it to something they know; start with their bodies, themselves, then go to manipulatives, then to pencil and paper.

This concern that students understand the meaning behind what they are doing, rather than just be able to produce answers, is one shared by many of the teachers, including EO, who says, "When we do problem solving, I'm not that concerned that they get the right answer, but more that they can say, 'Well, I had to add this and this together.' And I can say,

yes, you did! At least you're thinking the right way, and that's half the battle!" For EO, the workshop activities in strands other than arithmetic helped her to realize that knowing facts need not pre-date real problem solving, and that even her below-average math class could learn to think mathematically, rather than being stuck forever on low-level skills.

I was always trying to figure out ways to get them to know their facts, and how to regroup, and I never got much beyond that. And I never got into problem solving a whole lot, because they all had reading problems. It opened my eyes to how to get them thinking in math and doing much higher level math things than I ever thought they could. Plus it showed me there are a lot of ways to get at problem solving without requiring them to read in order to do it.

DC describes how her expectations have changed and how that has affected the attitude of her students. She strives for an "open-ended" attitude, "that there's no right answer sometimes", and that "what they see is just as valid as what I see, and their way can be just as good." As a result, "the kids have become more tenacious in their attempts to solve non-routine problems."

Another student outcome that is a result of placing emphasis on more than one type of math skill, and including activities from different strands as well as seeking and validating diverse approaches to problems, is that all students have an opportunity to feel successful. DC remarks on this change in the attitude of her students:

One of the things they feel is that they are all equal. In my classes in the past, there's been the top math group, the ones who felt they were better. There isn't any [top math group] in here, and that has made everyone more relaxed. No one assumes that they can do it better than anybody else.

LG refers to the Math a Way of Thinking approach as having a similar "equalizing effect" on her students. No one, she says, stands out as the "Math whiz", because "all approaches are valued, and the variety of tasks give different kids opportunities to show their strengths." This ability to feel successful is an important factor in developing a positive attitude toward one's ability to do mathematics.

Control. Another change teachers reported in their approach to teaching mathematics has to do with the issue of control. Many of their comments reflect an effort to move from the traditional authoritarian role of the teacher to that of a facilitator or guide in the process of sense-making and thinking. Part of this has to do with the kinds of questions teachers are asking, as discussed earlier; by asking more open-ended questions and validating students' thinking, teachers empower students, rather than reserving for themselves the role of passing judgement on what is right and what is wrong. Another aspect of giving up control has to do with allowing students to see that the teacher doesn't always know the answer or the outcome of an activity, and is learning along with his or her students.

DC sums up her teaching approach before the workshop when she says, "I was always the wise one imprinting the knowledge." She goes on to say that though she still falls into that role somewhat, she can now allow herself to be seen as fallible, and approaches more lessons as explorations and investigations for herself and her students together:

I feel comfortable enough with them [the materials] that at least I could admit to them 'Hey, I'm not real comfortable with this [meaning good at it], but let's see what we can do... I have allowed myself to be seen as ...'I don't know - let's find out.'

LK also talks about putting herself in the role of co-learner with her students, and the effect she sees on their attitude:

I'm learning and going through it with them. I think that has made a big difference. When I throw out questions, I'm also throwing them out for my own sake - I'd like to know, too. So I think just my enjoyment of what we're doing makes them enjoy it, too...They're not apprehensive and they're willing to throw out any questions they have about it.

RS describes how she feels less need to always be the leader or the one with the answers, and the benefits she sees for her students as a result of this change in her role.

I also find that I'm moving to the back of the classroom and they're moving to the front of the classroom a lot more often. The kids teach each other, and they're awfully good at doing that, and offering other ways to solve problems. I rely a lot on their strengths, because a lot of my kids are more mathematically talented than I will ever be, and they come up with some nifty suggestions for how to do things...I'm not giving up my position as teacher, but opening up to a more positive atmosphere, you know. The kids feel good about it, too. Sometimes that kid-to-kid makes a better, more meaningful experience than adult-to-kid.

One reason that teachers have felt a need in the past to maintain tight control over math lessons was because they had so little confidence in their own mathematical understanding. Their textbook became the framework for this rigid approach because it provided security - not only lesson plans, but answers. Several of the teachers interviewed reported that as their confidence and understanding has grown, they have become less dependent on the textbook and the control that it represents, and more inclined to spend more time on hands on and investigative learning. LC refers to how the workshop has made her work to change her attitude about teaching math, "to be more flexible and not so dependent on the paper work and the book work, and get out into those hands-on things."

The effort required to make this change, and the factors that help to bring it about, are summed up in this remark from SC:

I did it too - assign 30 examples on a page and that's your homework. It took me time...to want to give up that hold on, 'But I know they know it if they can do these,' to 'If they can do five of them they still know it.' You have to be ready to take a risk, you have to have the confidence to try it, to believe that there's another way out there to do it, and it's going to work.

The workshop experience, by immersing teachers in a different framework for learning mathematics, gives many of them the new beliefs and the confidence to take that risk.

Teaching as Thinking. DC reports that her whole focus has shifted from following the sequence and schedule set up by

her mathematics textbook, to following her observations and assessments of what her students understand.

I know in the past what I have introduced to kids. I don't think I'll get there this year, because I'm going to give them different things now after having the course [workshop]. I have a different philosophy now of what I want them to experience. Now, when they leave, I want to be sure that they really understand regrouping. Where before, I taught it to them and hoped they did.

This comment indicates another kind of change beginning to occur for some of the teachers I interviewed. They report becoming more critical of previously used methods and materials, and more likely to use their judgement in terms of what they think is appropriate to teach their students, and how to teach it. Their comments indicate that they are more tuned in to whether students are understanding something, so that expectations, approaches and activities can be changed as needed. They no longer want their students to learn by rote, and they are realizing that, as part of the process of teaching students to think, they can no longer "teach by rote" - that is, by following a prescribed set of lessons over a prescribed length of time. They are becoming thinking teachers, analyzing, diagnosing, and problem solving their way toward better mathematical understanding for their students.

BP refers to this as using her "instincts": "I've realized that if you don't slow down and do what they really don't know then it's not going to mean anything...I'm trusting my instincts more in terms of pacing and content."

I think she is referring to her new ability to judge what students understand, and their readiness for the next step, based on what they communicate through discussions and on what she observes them doing as they work with manipulatives. DC also talks about being able to "see" or analyze students' understanding in this way, and how her ability to assess this understanding more accurately has given her the confidence to make judgements about her mathematics program in general:

I can tell now whether they understand [place value] or not, through the manipulatives, and I'll stick with it until I'm sure everyone has it. Before, that chapter may have taken me three to four weeks; I can spend six to eight weeks on it now, and I'll be comfortable doing that. I'm much more willing to skip sections in the textbook, because I know now it's not important to do every page. I'm making judgements. They don't need to do a page on the commutative property, because we've talked about it so much with the arrays, and they know it. I'm much more confident in what I think is important for them to know, rather than getting through a certain curriculum.

RS also talks about how she feels more able to observe students reactions to activities in order to judge the level of their understanding, and more confident about selecting appropriate materials and activities:

I constantly look at my textbook and materials from a much more critical point of view, because I have a new frame of reference after taking the workshop...I look to see if it [a lesson] is child appropriate. The textbook I have now, for example, is not geared to children. The format of the teaching page is so complicated. I don't even use it anymore, whereas before I would have stayed with it. I use the concept, and I try to find other ways to present it. I look to see if I think the lesson is appropriate and if it isn't then I try to supplement it with material I have from Math a Way of Thinking.

And I really am cutting down on the number of problems I assign and I'm trying to glean from the kids and the five or six problems they've done what their errors are and we go back and figure out how to correct them.

EO describes how the place value lessons with beans and cups made "light bulbs go on" for her students, and how, by contrast, she could tell when a lesson was not making sense and she needed to rethink what she was teaching:

When you make a leap and they don't follow you, you can just see it in their eyes, and say, 'Whoa! We'll go back and try this again!' I had done that a couple of weeks ago. I thought we had done division enough that we could do it without manipulatives, and suddenly they had no idea what they were doing. I had thought, alright, enough of that, we'll go on... but it didn't work that way. Just because I think we're ready to go on doesn't mean they are.

Giving teachers this sense of their own ability to judge students' understanding and teach accordingly, rather than following a scripted textbook lesson, is an important outcome of the workshop for some teachers. The activities and the teaching style that are presented, the growth that teachers feel in their own ability to understand and do mathematics, and the flexibility that is encouraged, enables them to become less book-bound, and more open to picking and choosing what they think works best for their students. LK reflects this when she says:

Now that I have a better idea of what I'm doing and where I want to go with it, I'm able to pull from other math books and resources. Math a Way of Thinking is more a whole idea of math and teaching...It's really made a difference in what I'm doing.

EG is a teacher who had taught for many years through the traditional approach. She reports that she had always liked math because it was so "neat and orderly", and that she had always had trouble understanding why many students had trouble understanding something that was so "simply factual". As a result of her experience in the workshop, she realized that not all students saw or thought about things the same way that she did, and that she needed to be sensitive to this, and to adapt her teaching to suit the needs of the students:

I'm a very sequential person, but I have gotten less rigid about my teaching methods. I'm getting better at going from one to six sometimes, instead of always going from one to two to three to four to five to six... That's a positive change, because there are plenty of kids that work that way [not so sequentially]. The workshop has given me tools - I know now how to rephrase things, how to show them from a different angle.

Conclusion

The Mathematics a Way of Thinking workshop incorporates many of the factors that were established in Chapter III as being necessary for bringing about needed changes in how elementary school teachers teach mathematics. Teachers are working within a supportive environment in which they begin to rely on their own thinking to help them make sense of mathematical ideas and concepts. Because of group work and class discussions, teachers are exposed to a wide range of ways of seeing and thinking about mathematical situations. Often for the first time, teachers see a creative, open-ended

side of mathematics, its connection to other curriculum areas, enjoy the process of doing mathematics, and experience increased confidence in their own ability to do mathematics. They are exposed to a broad spectrum of methods, sample activities, and materials that can help them to teach math themselves in a more exciting and meaningful way.

In one week the workshop cannot teach a great deal of mathematics content, and that is not its purpose. Its purpose is primarily to provide an experience through which teachers can change what they think and how they feel about mathematics, and begin to see the benefits for their students of learning through a thinking, active learning approach. Positive dispositions toward mathematics and toward teaching mathematics must be in place before teachers will consider changing how they teach it. Once teachers believe that mathematics is an important and necessary skill, and feel confident in their ability to do mathematics and to teach it effectively, then they are ready for further study of mathematics concepts and applications.

Comments from those teachers I interviewed indicate to me that the workshop does begin this process for many teachers. It improves their understanding of mathematics concepts, their feelings about doing mathematics, and their skills for teaching mathematics. It helps to bring about insights into what mathematics is, and how it is learned. EO

summed up the experience of many teachers in her comment, "I feel as though I've just figured out how to teach math."

I would like to end this chapter with a note written to me by EG shortly after my interview with her, in which she sent me her further thoughts on the questions we had discussed in the interview.

It occurred to me that 'Math a Way of Thinking' was the best of all possible titles for this course. I say this because your course, more than any other workshop, has had an impact on the way I 'think' about math. It has changed the way I view all math students - especially those who fear math and view themselves as poor math students.

Never before have I seen so clearly the need for using concrete, hands-on materials. Never had I realized the importance of laying the groundwork for new concepts using actual objects in real situations.

The other change occurred in my attitude toward the question and answer part of my math classes. My policy had always been that all classes should be closed up in neat little packages - all questions answered, all problems solved. I now feel that my lessons are not complete unless all children walk away thinking, 'I wonder what would happen if...'

The Mathematics a Way of Thinking Workshop also immerses teachers in a model of a classroom in which critical thinking is "at the heart of instruction" (NCTM 1989, p. 29).

Throughout the week of the workshop, teachers are learning and re-learning math concepts within an investigative, problem solving context, using concrete materials and active learning. The instructor models teaching approaches that encourage inquiry and reasoning, the two areas of critical thinking skills defined in Chapter II. To illustrate the

relationship of the more general critical thinking skills to specific goals for mathematics education as described by NCTM, and the Mathematics a Way of Thinking workshop, a table is provided in Appendix B.

C H A P T E R V I

ISSUES AND REFLECTIONS

I have no doubt that the Mathematics a Way of Thinking workshop makes an impact, and sometimes a very powerful impact, on many of the participants. It brings about personal insights into mathematical concepts, and the structure of mathematics, and positive changes in attitudes towards doing and teaching mathematics. Many participants, on their final evaluation cards, write that they are excited about mathematics, and about making it more meaningful and exciting to their students. As one participant wrote, "I have learned the meanings behind the motions."

However, a week of training, no matter how intense and effective, can only be a beginning. No matter how much teachers feel changed personally by their experience in the workshop, they must still deal with the realities of attempting to put their new insights and knowledge into practice in the classroom. Teachers have a great deal of autonomy within their own classrooms, but they still must answer to organizational aspects of their school systems, which can, and sometimes do, stifle teachers' attempts to improve their instruction. The NCTM, in Professional Standards for Teaching Mathematics (1991), makes a basic assumption that "Changing the practices of mathematics teaching depends on teachers, but teachers cannot

effect...reform without substantial systematic support and change" (p. 3).

This valid issue of support from the system comes up in the workshop in the form of questions raised by teachers as they contemplate implementing what they are learning within their own classrooms. Their questions have to do with "covering" the curriculum, testing schedules, expectations of administration, and parental expectations. Often teachers questions reflect their frustrations over the conflict they perceive between what they want to do, and what they feel required to do. These are valid questions about issues that must be resolved if change is to occur.

The Curriculum

In many school systems, the curriculum is a traditional textbook, based on some anonymous standard setter's ideas of what children ought to know. A common question from teachers is, "How can I do this and finish the curriculum, too?" Many teachers feel pressured to complete the textbook, partly because they have always felt insecure about teaching mathematics, and the book provides a secure framework; they recognize the time required to teach investigatively, and are worried that they won't "cover" enough material. Another source of pressure to complete the textbook is often the building principal.

Conflict arises from teachers' newly forming beliefs about how students should learn mathematics, and these imposed expectations. Teaching for understanding and thinking does not happen in neat packages or fit a predictable schedule; a textbook often gives specific lessons, including questions to be asked, answers to expect, and a time limit for each lesson. Conflict also comes about because teachers sometimes receive mixed messages; they will be encouraged to take the workshop, and to implement the approach, but still be expected to "cover the curriculum". This conflict can discourage those who are making a first attempt at change.

School districts have rather specific expectations for the amount of material to be completed in mathematics at each grade level. Curricula are often keyed to particular objectives and standardized tests to ensure coverage and mastery. Although teachers may initially attempt to teach for conceptual understanding, after a relatively brief period of developing instruction using models, for example...they feel compelled by time pressures to complete the 'coverage' of the topic. (Hyde 1989, p. 224)

Teachers are not accustomed to bucking the tide. They have historically accepted the curriculum handed to them, and struggled to imprint it upon their students. In the workshop, I encourage teachers to become more outspoken about the mixed messages they are receiving. As they become more confident in their ideas about how mathematics is learned, they need to become active in changing the curricular expectations within their systems. As they have begun to

learn how to judge their students' understanding and vary their teaching accordingly, so must they begin to judge the worth of what they are being asked to teach.

Each topic allotted time in the curriculum must be justified on the basis of the role it plays in the students' overall mathematical growth...Time [for a more investigative approach] can be found by reducing the time previously spent on over-practicing computational procedures. (Dossey 1989, p. 24)

Testing

Most teachers in the workshop, as a result of their own immersion in the process, recognize the worth of teaching mathematics as an investigative, thinking process. But inevitably, questions arise about testing, usually standardized testing: "How will my kids do on the test?" or "Will they be ready for the test in April?". Though teachers may value the thinking approach to teaching, many of them are judged, literally, according to how their students perform on standardized tests, which test lower level skills and heavily stress computation. As Schoenfeld notes, "In general, 'having an ability' has been defined as scoring well on a test for that ability" (1989, p. 8).

This creates another conflict for teachers who value teaching their students to think mathematically. If the system recognizes ability and rewards it based on a test that measures something other than what the teacher is teaching, then the pressure is very strong to go back to the

traditional way of teaching. This is particularly true if the teacher's attempts to change her or his teaching are in the beginning stages, and still shaky. Some teachers' sense of conflict is doubled because they may again receive a mixed message from their system: implement this approach, but we still expect your test scores to be high.

Teachers need to be reassured and given evidence that assessment methods are also in the process of change, and that even on standardized tests, students who learn through an active learning approach do well over time. Lindquist states: "The short term pay-off for students knowing 'what to do' is great because that is what we reward [through standardized tests]. The long-term payoff is a disaster, as shown by the present state of mathematical learning." I urge teachers to become advocates for the long-term changes that must come to mathematics education, and to question, loudly, the testing policies of their system and the purposes for which standardized tests are used.

Administration

Out of all the workshops I have taught, I have had only one participant who was a building principal. Yet principals are responsible for overseeing the teaching that is going on within their buildings, and are part of the force behind testing and curriculum expectations. Teachers planning to attempt changes in their mathematics teaching style and

content often express a concern that their principal will not understand what they are doing, or will even discourage or forbid many activities, because they are noisy, or aren't part of "covering the book". The conflict here is between teachers who have sought to improve their professional abilities, and the learning of their students, ensnared by a principal who operates in the dark ages, who has what Wiggins calls an "essentially medieval view of curriculum, premised on the finite and static quality of knowledge" (1989, p. 45).

Administrators must also be re-educated to understand the need to strike a balance between developing appropriate concepts and reasoning, and procedural knowledge. As Hyde notes, "The building principal is traditionally more concerned with teacher evaluation than with instructional improvement" (1989, p.224). The question that must be asked is the following: What worthwhile set of criteria for evaluation of teachers is the principal using, if not their ability to improve instruction? I constantly urge administrators to take the workshop, or some other similar training, so that they will be able to judge whether or not their staff are implementing pedagogically and mathematically sound innovations.

The NCTM (1991) is clear about what administration should do in order to help bring mathematics teaching out of the dark ages. They should (1) implement staff development programs, (2) involve teachers in designing and implementing

such programs, (3) provide adequate resources, including time and funding, to effectively implement such programs, and (4) promote collegiality through involving all teachers in such staff development. In other words, they should act as an informed support system which nurtures a professional teaching staff.

Parent Expectations

Because parents all once went to school, they often see themselves as educational experts. Sometimes they are alarmed by practices that are different from what they remember from their school experience. Many of them are also overly concerned with the results of standardized tests. In the workshop, teachers often ask, "How do parents react to this?" or otherwise comment that they are concerned with parents wanting to see the worksheets coming home so that they know "math" is happening. In these questions, I hear the very real concern about the conflict that can arise between a professional who knows that her or his teaching practices are effective, and a parent who thinks they know what is right, based on their own school experience and lack of information.

Teachers who become advocates for changing mathematics education must accept the fact that part of their job will be to re-educate parents as well as their students. I suggest that through individual parent conferences and newsletters,

and by inviting parents to join mathematics classes or special parent "math nights", parents can be helped to understand the purposes and the benefits of the changes being made. Once they realize that their children are learning, and learning with understanding, they are generally supportive.

All of these are valid questions and concerns that challenge even a devoted teacher. Change must begin with teachers, but it can also end with them without the support of the system.

What's Next?

Another issue of support, other than systematic support, that requires serious consideration once the initial workshop experience is past, is what comes next in order for teachers to sustain an effective implementation of the teaching approach and curricular changes called for by the NCTM and modeled in the Mathematics a Way of Thinking workshop. I have identified four factors that need to be considered: (1) continuing training in mathematics content; (2) more specific training in teaching thinking skills; (3) structures that foster a sense of collegiality and common goals; and, (4) becoming comfortable with the time needed to bring about real change.

Knowledge of Mathematics. The workshop helps to free participants from old math beliefs and anxieties, and clarifies some very basic concepts and connecting threads in mathematics, but it is only a beginning. Teachers, for the most part, leave the workshop excited and armed with activities that they have proved on themselves and feel fairly confident about trying with their students.

However, it would be naive to assume that teachers can automatically transfer the teaching techniques or the philosophy of the workshop to everything they teach in mathematics. Teachers cannot relearn all that they need of mathematics and teaching mathematics in a week. What the workshop serves to do, I think, is to open the way for teachers to learn more mathematics. By beginning to change belief systems about mathematics, and by bringing about more positive dispositions toward mathematics, the workshop provides a framework within which teachers can begin to see the sense and the value of learning mathematics. And, the more thorough a grounding teachers have in mathematics content, the more able they are to select appropriate tasks for their students and help them to see the connections within mathematical principles and ideas (NCTM 1991).

Without further training, I have a concern that teachers leave the workshop excited by their own learning, with a set of activities that they know will "work", but without a firm enough understanding of mathematics to develop other lessons

or seek out other sources of information for themselves. Ongoing mathematics training needs to be readily available, so that teachers can continue to deepen their understanding of concepts and processes. This will help to prevent a "bag of tricks" approach to the workshop and similar types of short term training, in which the activities are used as fillers or "Fun Friday" activities, while the textbook remains the primary source of instructional material. The workshop activities and approach must be seen as part of an integrated approach to teaching mathematics, not as an add-on.

Teaching Thinking. The workshop immerses teachers in an environment in which the activities themselves and the questions and responses of the instructor bring about thinking. It exposes teachers to the effects of being part of a classroom in which teaching for thinking is going on as a matter of course. Part of the stimulation that teachers feel as a result of the workshop is due to their own engagement in using critical thinking skills in the process of making sense of mathematics. However, in the workshop, time is not spent on identifying the skills that are being used, or on discussing ways to ensure that they are infused into lessons. The instructor simply models teaching for thinking, and engages participants in the process.

As I have stressed before, the workshop is a beginning. Though the time frame and schedule do not allow for direct teaching of thinking skills, through engaging teachers in using these skills, the value of teaching for thinking becomes apparent. And, as established in Chapter 3, the value of teaching something must be recognized, in a personal sense, before teachers will change their teaching to include it. The workshop experience helps to make teachers receptive to further training, in which specific thinking skills and techniques for infusing them into lessons would be the emphasis.

Sense of Collegiality. The workshop gives teachers a sense of working toward a common goal. At first, they are there as separate entities to gather knowledge to take back and impart to their students. But the structure of the workshop creates the necessity for them to help each other gain that knowledge. As a result of working together to learn mathematics concepts and teaching approaches, and sharing common concerns, they become united by the common goal of improving their teaching and their students' learning. By the end of the week there is a sense of shared purpose and support for each other, and of shared needs. By Friday, it is not uncommon for participants to openly share their confusion about something we are doing, to laugh at their mistakes or misconceptions, and to applaud a participant who

has done something that the others know is difficult for him or her. This sense of collegiality adds greatly to participants' abilities to envision themselves changing their teaching practices, and serves also as a model for the environment that can develop in a classroom. Often participants comment that they hate to see the week end, because they will miss the group process.

In order to help teachers implement changes in how mathematics is taught, there need to be structures in place that provide ongoing "supportive interaction" (Hyde 1989) among teachers. Teachers need to feel, as one participant put it, that they are "all in the same boat".

To relinquish established systems of thought and action, a person has to...get assistance from sympathetic and supportive others in making the transition to new modes...Teachers need to realize that their feelings about teaching mathematics are not unique. They need nonevaluative assistance and reassurance from leaders and their peers that they can overcome difficulties and develop more effective teaching strategies. (Hyde 1989, p. 227)

For some participants this need is filled for a time by the follow-up meetings. I offer them once a month for six months, and they meet for three hours each time. They have several purposes. We review activities from the workshop, and explore extensions of them. We explore new activities and problem solving situations. We share successes and failures, doubts and concerns, ideas and frustrations. I think that above all, the sessions provide a collegial structure that helps people to continue doing the hard work

of changing old habits. They may no longer be comfortable habits, but it is still hard work to change them. One participant called the sessions her "monthly therapy sessions", and many have commented that without the follow-ups they would have implemented far less of the workshop approach in their classrooms throughout the year.

Without a support group, the effort to change in isolation is often too doubt-ridden and overwhelming to succeed. Ideally, such supportive peer groups should be in place in any school that is trying to bring about change in how teachers teach mathematics.

Time for Change. By about the third or fourth day of the workshop, participants begin to make anxious comments such as, "How am I ever going to remember how to do all this?" I take time at that point to assure teachers that they won't remember it all, and that they shouldn't worry about trying to remember or do it all "at once". Though teachers recognize that their students often need to do things or hear things many times before they remember them or make sense of them, they somehow expect that once should be enough for them.

What teachers in the workshop are feeling, I think, is partly a result of their own excitement about what they are learning and their own sense of urgency about wanting to use it with students. But it is partly a result of dealing with

unrealistic or impatient administrators. In my experience, teachers have often been given a new curriculum or textbook and a minimum of training and then been expected to fully implement a new program practically overnight, without proper consideration given to the time needed for the process of change.

Sobel, commenting on past attempts at changing mathematics education, says, "By now it should be apparent that change is a complex process that comes about in slow stages" (1981, p. 189). That this is so is backed by many studies (Hyde 1989; Lieberman and Miller 1981). Changing how teachers teach is complex and slow because it involves changing deep rooted factors, as discussed in Chapter 3, such as knowledge, beliefs, long established patterns of thinking and habits of doing.

When I tell teachers that they should not expect, or even try, to remember everything we do, they seem surprised. I go on to encourage them not to be too hard on themselves in terms of what they set as goals for their first year of trying to change their mathematics teaching. I suggest that they choose one or two things from the workshop with which they are comfortable, and try them, then build on that the next year. I suggest that they think in terms of a three to five year plan for changing how they teach mathematics, and that they see it as a matter of constant growth, adding more to their repertoire as they become more secure about what

they know and about using the process. I know that some of them will make changes quickly, and others will proceed with great caution. I point out that, though I am the instructor for the workshop, I am still in the process of relearning mathematics, and still need to find more effective ways to teach many concepts. Real meaningful change happens slowly over time, not in one fell swoop, and teachers need to know that they are not working against an artificial time limit.

Reflections on My Own Growth

In Professional Standards for Teaching Mathematics, the NCTM states the following assumption:

Teachers are in a constant state of 'becoming'; that is, being a teacher implies a dynamic and continuous process of growth that spans a career. (1991, p. 63)

The idea of being in a "dynamic and continuous state of growth" certainly applies to me, especially over the course of the past several years. It was sparked by my conviction that there had to be a better way to teach mathematics, by an almost fierce dissatisfaction with the way things were. My experience as a participant in the Mathematics a Way of Thinking workshop provided a framework for understanding how better to teach mathematics. And my work in the Critical and Creative Thinking program provided insights into the rich possibilities for teaching thinking within that framework.

However, my understanding of the teaching and learning process has been greatly deepened by teaching other teachers,

by trying to help them understand something which becomes ever more clear to me, by coaching and coaxing them to give up old fears, old beliefs, and old expectations of themselves and their students. Through their questions, I realize more and more the complexity of the attempt to change not only themselves, but a whole dinosaur-like system. I celebrate every 'light bulb' that goes on, and the potential change that it represents in some student's mathematics learning experience.

As I work with teachers, it also becomes more clear that, just as children move through developmental stages in their understanding, so can teachers, given the right interventions. Hyde (1989) talks about the developmental stages of teacher change, and the fact that it must be viewed as vertical, not horizontal, change. In other words, in order to change how teachers think about mathematics, new ideas must be processed down (or up?) through a teacher's pre-existing framework for making sense of the world; they cannot be simply transferred or imprinted, full blown, into a teacher's repertoire. New ideas or experiences that would bring about change must also be developmentally appropriate - they must fit the framework of the target audience or they will not be incorporated into that framework.

Like the view held by cognitive psychology that mathematical concepts and meanings must be constructed by each child from his or her experiences, teachers must

construct, or re-construct, their conceptual framework of mathematics and of teaching before that can change, in a meaningful way, how they teach mathematics.

According to the constructivist perspective, we all build our own interpretative frameworks for making sense of the world, and we then see the world in the light of these frameworks. (Schoenfeld 1987b, p. 22)

It follows that as we incorporate new information into this framework, the framework changes slightly to accomodate the new information, so that the next piece of information is understood, or made sense of, in a slightly different way than it would have been previously. Also, because we each have constructed our own frameworks from a different array of experiences, a new piece of information encountered by myself and someone else simultaneously will be interpreted and incorporated into our respective frameworks differently for each of us.

Let me apply this idea to my own development, starting with the first time that I was a participant in the Mathematics a Way of Thinking workshop. I was excited by the hands on materials, especially the place value work with beans and cups. I shared the typical elementary teacher's obsession with teaching regrouping to students who stubbornly refused to learn it. I focused on these activities; they fit what I was willing to take into my framework. I was concerned with the fact that they worked, and with replicating them in my classroom. The rest of the workshop

was enjoyable for the most part, although I tossed aside the non-routine problem solving homework assignments, irritated that I couldn't do them, but thinking they were not "math" anyway.

That year I used the place value activities very effectively. I also used other activities from the workshop, but more with a "bag of tricks" approach than with any real understanding of their mathematical significance. My focus was still very much on computation, albeit now through manipulatives.

The next summer I took the workshop again. My experimentation with the activities throughout the year, and my continuing course work in the Critical and Creative Thinking program had enlarged my "framework". This time I became more interested in activities I couldn't even remember from the summer before. I was more intrigued by the neglected strands in mathematics, by the thinking skills being used, and incredibly aware of the holes in my own mathematics education. This time I listened to the discussions about solutions to the homework problems with new ears, and came to the realization that my "framework" did not include strategies for approaching them, or even an inkling as to why they were important. This time I was concerned more with why the activities worked, and how I could begin to adapt what worked to other concepts. I couldn't think of many adaptations; I was still very much stuck in the habits

of mind that came from traditional mathematics learning, and needed lots of help visualizing concrete representations for things I had only learned abstractly. But with each new application I learned, the framework enlarged, as well as the desire to enlarge it.

Things make sense to me now, can be incorporated into my "framework", that I know I would have simply ignored a few years ago. And I realize the importance of never thinking that that framework is finished; I am a teacher, but I am above all a learner. I take to heart DeFelice's belief that "As perpetual experts, teachers are shackled; as learners, they are set free" (1989, p. 641).

It is important that I apply what I now understand about frameworks to my work with teachers. When I lead a workshop, I cannot assume that I know the developmental stage, in terms of readiness to receive what I have to offer, of the teachers before me. I cannot assume that I know what they will take away with them, in terms of knowledge or perception of what is important. They all come with a different framework, and I can only know that their frameworks are different from mine. I must be cautious, in my zeal, not to try to pull them "up" through levels of understanding to what I have come to understand. I must try, always, to go back to the beginner's framework, to provide experiences and questions in such a way that the teachers begin their own "vertical transfer" and process of growth.

Duckworth might be speaking of the Mathematics a Way of Thinking workshop when she says:

We encouraged teachers to take their own knowledge seriously, to be willing to pay attention to their confusion, to make an effort to understand each other's ways of understanding the phenomena, to take the risk of offering ideas of which they were not sure. (1987, p. 84)

It is through this process of dealing with their own uncertainty and constructing their own meaning that teachers develop richer frameworks that enable them to cast off the shackles of traditional teaching and become free to facilitate children's learning.

I see my role as one of presenting situations in which this can happen, asking questions that facilitate the process, and providing support, encouragement and reassurance as teachers struggle to incorporate what they can into their respective frameworks. I find that reassurance is needed that:

- change, even when you want it, happens slowly and not without discomfort, and can't be rushed. Each of us must do what feels comfortable and possible for us.

- things don't always work out as planned or envisioned. Something modeled in the workshop may flop the first time you do it in your classroom. Learn from the flops, and try again. Be prepared for unexpected outcomes, and use the unexpected as a learning tool.

- research supports what you are doing.

-don't be afraid to say, "I don't know". Be a co-learner with your students; learn with them and from them. You don't have to know all the answers to try new things. As DC said, "Everyday I learn something new from teaching this way."

Another important way that I can support participants' efforts to change is by sharing with them the process of my own growth. I remind them that I once sat where they sit, somewhat baffled and unable to take it all in. I share my own frustrations, such as the homework story, my own math anxieties, and my own failures when I began, as well as my success stories about student outcomes. I stress the fact that, though I am the instructor, I am still very much a learner, and that I come away with a deeper understanding of what we are doing with each workshop that I teach. The message, I hope, is that this process doesn't have to do with the traditional idea of the teacher becoming the expert; it has to do with an ongoing process of learning so that we can continue to improve our ability to teach.

I also, more and more, urge teachers to become better informed about the research behind this approach to teaching mathematics, and less passive in the face of systematic stumbling blocks. I urge them to read the Standards, and professional journals, and to arm themselves with information in order to better deal with those who question what they are doing. It is one thing to say, "I'm doing this because I feel it's right"; it's another to be able to cite real

sources that support what you are doing. My goal is not only to introduce teachers to a new approach to teaching mathematics, but to make them advocates of the change process.

I do not harbor any fantasies that the workshop makes a major impact on education in general. As Suzanne Wilson says of working with teachers:

In ten weeks, I can't hope to pull out all the deeply rooted beliefs of my students...but in one term I can shake them up a little and help them begin to examine their assumptions about learning and teaching. I can also provide a safe environment in which they can begin to act like real learners - challenging and justifying, hypothesizing and experimenting. (1990, p. 208)

I have only one week. Although it is an intense week, it cannot possibly root out all past beliefs and practices and replace them with new ones. However, for many teachers it provides a meaningful beginning, because it at least throws those past beliefs and practices into doubt, and gives them a taste of alternatives.

Since I became involved in the Mathematics a Way of Thinking program, I have felt strongly that the workshop makes an impact on many teachers. The work done for this thesis has provided evidence that the workshop does offer a significant experience in learning mathematics as a thinking, investigative, sense-making process, and that it can be a catalyst for real change for teachers.

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APPENDIX A

Goals for a Critical Thinking/Reasoning Curriculum

- I. Working definition: Critical thinking is reasonable reflective thinking that is focused on deciding what to believe or do.
- II. Critical thinking so defined involves both dispositions and abilities:
 - A. Dispositions
 - 1. Seek a clear statement of the thesis or question
 - 2. Seek reasons
 - 3. Try to be well informed
 - 4. Use and mention credible sources
 - 5. Take into account the total situation
 - 6. Try to remain relevant to the main point
 - 7. Keep in mind the original and/or basic concern
 - 8. Look for alternatives
 - 9. Be open-minded
 - a) Consider seriously other points of view than one's own (dialogical thinking)
 - b) Reason from premises with which one disagrees--without letting the disagreement interfere with one's reasoning (suppositional thinking)
 - c) Withhold judgement when the evidence and reasons are insufficient
 - 10. Take a position (and change a position) when the evidence and reasons are sufficient to do so
 - 11. Seek as much precision as the subject permits
 - 12. Deal in an orderly manner with the parts of a complex whole
 - 13. Use one's critical thinking abilities
 - 14. Be sensitive to the feelings, level of knowledge, and degree of sophistication of others
 - B. Abilities
 - 1. Focusing on a question
 - a) Identifying or formulating a question
 - b) Identifying or formulating criteria for judging possible answers
 - c) Keeping the situation in mind
 - 2. Analyzing arguments
 - a) Identifying conclusions
 - b) Identifying stated reasons
 - c) Identifying unstated reasons
 - d) Seeing similarities and differences
 - e) Identifying and handling irrelevance
 - f) Seeing the structure of an argument
 - g) Summarizing

3. Asking and answering questions of clarification and/or challenge, for example:
 - a) Why?
 - b) What is your main point?
 - c) What do you mean by "_____".
 - d) What would be an example?
 - e) What would not be an example (though close to being one)?
 - f) How does that apply to this case (describe a counterexample)?
 - g) What difference does it make?
 - h) What are the facts?
 - i) Is this what you are saying "_____".
 - j) Would you say some more about that?
4. Judging the credibility of a source
 - a) Expertise
 - b) Lack of conflict of interest
 - c) Agreement among sources
 - d) Reputation
 - e) Use of established procedures
 - f) Known risk to reputation
 - g) Ability to give reasons
 - h) Careful habits
5. Observing and judging observation reports: criteria:
 - a) Minimal inferring involved
 - b) Short time interval between observation and report
 - c) Report by observer, rather than someone else (i.e., not hearsay)
 - d) Records are generally desirable; if report is based on a record, it is generally best that
 - 1) The record was close in time to the observation
 - 2) The record was made by the observer
 - 3) The record was made by the reporter
 - 4) The statement was believed by the report, either because of a prior belief in its correctness or because of a belief that the observer was habitually correct
 - e) Corroboration
 - f) Possibility of corroboration
 - g) Conditions of good access
 - h) Competent employment of technology, if technology is useful
 - i) Satisfaction by observer (and reporter, if a different person) of credibility criteria (item B4)
6. Deducing and judging deductions
 - a) Class logic
 - b) Conditional logic
 - c) Interpretation of statements
 - 1) Double negative
 - 2) Necessary and sufficient conditions
 - 3) Other logical words and phrases: only, if and only if, or, some, unless, not, not both, etc.

7. Inducing and judging inductions
 - a) Generalizing
 - 1) Typicality of data
 - 2) Limitation of coverage
 - 3) Sampling
 - b) Inferring explanatory conclusions and hypotheses
 - 1) Types of explanatory conclusions and hypotheses
 - a) Casual claims
 - b) Claims about the beliefs and attitudes of people
 - c) Interpretations of authors' intended meanings
 - d) Historical claims that certain things happened
 - e) Reported definitions
 - f) Claims that something is an unstated reason or unstated
 - 2) Investigating
 - a) Designing experiments, including planning to control variables
 - b) Seeking evidence and counterevidence
 - c) Seeking other possible explanations
 - 3) Criteria: Given reasonable assumptions
 - a) The proposed conclusion would explain the evidence (essential)
 - b) The proposed conclusion is consistent with known facts (essential)
 - c) Competitive alternative conclusions are inconsistent with known facts (essential)
 - d) The proposed conclusion seems plausible (desirable)
8. Making value judgements
 - a) Background facts
 - b) Consequences
 - c) Prima facie application of acceptable principles
 - d) Considering alternatives
 - e) Balancing, weighing, and deciding
9. Defining terms, and judging definitions in three dimensions
 - a) Form
 - 1) Synonym
 - 2) Classification
 - 3) Range
 - 4) Equivalent expression
 - 5) Operational
 - 6) Example-nonexample
 - b) Definitional strategy
 - 1) Acts
 - a) Report a meaning (reported definition)
 - b) Stipulate a meaning (stipulative definition)
 - c) Express a position on an issue (positional, including programmatic and persuasive definition)
 - 2) Identifying and handling equivocation
 - a) Attention to the context
 - b) Possible types of response
 - i) The simplest response: "The definition is just wrong"
 - ii) Reduction to absurdity: "According to that definition, there is an outlandish result."

- iii) Considering alternative interpretations: "On this interpretation, there is this problem; on that interpretation, there is that problem."
 - iv) Establishing that there are two meanings of key term and a shift in meaning from one to the other
 - v) Swallowing the idiosyncratic definition
 - c) Content
- 10. Identifying assumptions
 - a) Unstated reasons
 - b) Needed assumptions; argument reconstruction
- 11. Deciding on an action
 - a) Define the problem
 - b) Select criteria to judge possible solutions
 - c) Formulate alternative solutions
 - d) Tentatively decide what to do
 - e) Review, taking into account the total situation, and decide
 - f) Monitor the implementation
- 12. Interacting with others
 - a) Employing and reacting to fallacy labels, including
 - 1) Circularity
 - 2) Appeal to authority
 - 3) Bandwagon
 - 4) Glittering term
 - 5) Name calling
 - 6) Slippery slope
 - 7) Post hoc
 - 8) Non sequitur
 - 9) Ad hominem
 - 10) Affirming the consequent
 - 11) Denying the antecedent
 - 12) Conversion
 - 13) Begging the question
 - 14) Either--or
 - 15) Vagueness
 - 16) Equivocation
 - 17) Straw person
 - 18) Appeal to tradition
 - 19) Argument from analogy
 - 20) Hypothetical question
 - 21) Oversimplification
 - 22) Irrelevance
 - b) Logical strategies
 - c) Rhetorical strategies
 - d) Argumentation; Presenting a position, oral or written
 - 1) Aiming at a particular audience and keeping it in mind
 - 2) Organizing (common type: main point; clarification; reasons; alternatives' attempt to rebut prospective challenges; summary, including repeat of main point)

This is only an overall content outline. It does not incorporate suggestions for level, sequence, repetition, greater depth, emphasis, or infusion in subject matter area (which might be either exclusive or overlapping). Ennis (1987)

APPENDIX B

Critical Thinking Skills for Mathematics

Critical Thinking Skills (Ennis 1987) Skills of Inquiry	NCTM Standards (1989) with Examples from Mathematics a Way of Thinking Workshop
1. Identifying and formulating questions.	<p>Students should be involved extensively in exploration of problem situations "rich in opportunities to formulate and define problems" (p. 67).</p> <p><u>Workshop Examples:</u> Probability activity. What would happen if we used three dice?</p>
2. Asking and answering questions of clarification and challenge.	<p>Students should be asked questions that require them to justify their answers and their thinking, and should learn to ask such questions themselves.</p> <p><u>Workshop Examples:</u> Throughout the workshop activities, the instructor asks, "Why do think so?" Participants become increasingly more interested in the process than in solutions.</p>
3. Investigating. Collecting facts, evidence, or explanation to support a conjecture; searching for facts or information to solve a problem.	<p>Mathematics should be approached as a problem-solving process which stresses a "method of inquiry and investigation" (p.75).</p> <p><u>Workshop Examples:</u> Workshop participants collect data and search for a pattern in the process of discovering the formula for the area of a triangle.</p>
Critical Thinking Skills Skills of Investigation	NCTM Standards (1989) with Examples from Mathematics a Way of Thinking Workshop.
4. Analyzing arguments. Identifying conclusions, determining validity of conclusions, establishing and testing criteria.	<p>The mathematics curriculum should allow time and experiences for students to "develop their ability to construct valid arguments...and evaluate the arguments of others" (p. 81).</p> <p><u>Workshop Examples:</u> Throughout the workshop, participants share their thinking and discuss strategies and solutions for problems.</p>
5. Deducing and judging deductions. Is a conclusion consistent with all the facts collected?	<p>The study of mathematics should emphasize reasoning so that students can draw logical conclusions about mathematics, and apply deductive reasoning.</p> <p><u>Workshop Examples:</u> Workshop participants use reasoning skills in place value activities, and when finding areas of various triangles on the geoboard. Homework assignments emphasize deductive reasoning.</p>
6. Inducing and judging inductions. Generalizing and inferring conclusions and hypotheses, often based on discovery of a pattern.	<p>Students should learn to recognize and apply inductive reasoning, make and evaluate mathematical conjectures, generalize solutions and strategies, and recognize, describe, and generalize patterns.</p> <p><u>Workshop Examples:</u> The inductive approach is applied throughout the workshop. Participants are constantly asked to look for and describe patterns, and to generalize conclusions or solutions based on the patterns they find.</p>

Critical Thinking Skills General Skills	NCTM Standards (1989) with Examples from Mathematics a Way of Thinking Workshop
7. Defining terms. Establishing a common understanding of concepts and terminology.	<p>Communication is an important aspect of learning to think mathematically. Students must "reach agreement about the meanings of words and recognize the crucial importance of commonly shared definitions" (p. 78).</p> <p><u>Workshop Examples:</u> Throughout the workshop, the instructor frequently emphasized the importance of defining mathematical terms and making sure that students and teacher share a common language.</p>
8. Interacting with others in discussions, presentations, and debates.	<p>Students must have numerous opportunities for communicating about mathematics. Through small group problem solving and sharing of ideas, students learn to clarify their own thinking. Such interaction serves to "stimulate deeper understanding of concepts and principles" (p. 78).</p> <p><u>Workshop Examples:</u> Workshop participants work with partners or small groups, sharing ideas and strategies and developing a broader framework for thinking about and doing mathematics.</p>